

# Dynamical Symmetry Breaking in Chiral Gauge Theories with Direct-Product Gauge Groups

Yan-Liang Shi (石炎亮) and Robert Shrock

*C. N. Yang Institute for Theoretical Physics, Stony Brook University, Stony Brook, N. Y. 11794*

We analyze patterns of dynamical symmetry breaking in strongly coupled chiral gauge theories with direct-product gauge groups  $G$ . If the gauge coupling for a factor group  $G_i \subset G$  becomes sufficiently strong, it can produce bilinear fermion condensates that break the  $G_i$  symmetry itself and/or break other gauge symmetries  $G_j \subset G$ . Our comparative study of a number of strongly coupled direct-product chiral gauge theories elucidates how the patterns of symmetry breaking depend on the structure of  $G$  and on the relative sizes of the gauge couplings corresponding to factor groups in the direct product.

## I. INTRODUCTION

A problem of longstanding interest has been the behavior of strongly coupled chiral gauge theories (in four spacetime dimensions, at zero temperature). Here a chiral gauge theory is defined as one in which the fermions, written in left-handed chiral form, transform as complex representations of the gauge group. A chiral gauge theory is defined as being irreducibly chiral if it does not contain any vectorlike subsector. In this case, the chiral gauge symmetry forbids any fermion mass terms in the underlying Lagrangian. In order for the theory to be renormalizable, one requires that it must be free of any triangle anomalies in gauged currents.

In this paper we shall analyze a variety of chiral gauge theories with direct-product gauge groups of the form

$$G = \bigotimes_{i=1}^{N_G} G_i \quad (1.1)$$

with fermion contents chosen so that all non-Abelian gauge interactions are asymptotically free. The reason for this choice is that this enables one to carry out perturbative calculations at a sufficiently large Euclidean energy/momentum scale,  $\mu$ , in the deep ultraviolet (UV). As the theory evolves from the UV to the infrared (IR), these non-Abelian gauge interactions thus grow in strength. We restrict here to theories without fundamental scalar fields. The gauge group  $G$  is taken to contain  $N_{NA}$  non-Abelian factor groups, and, by convention, we order the factor groups in the tensor product (1.1) so that these non-Abelian factor groups come before any possible Abelian factor group(s).

The main question that we investigate is how patterns of dynamical gauge symmetry breaking depend on the structure of the direct product gauge group (1.1) and on the relative strengths of the gauge couplings for various factor groups  $G_i \subset G$  that become strong in the IR. We assume that if  $G$  contains any Abelian gauge interaction, it is weakly coupled at high scales  $\mu$  in the UV; given that such a gauge interaction has a positive beta function, this implies that the Abelian coupling will also remain weak at lower scales in the infrared. Our study of a variety of direct-product chiral gauge theories shows how the pat-

terns of symmetry breaking depend on the structure of  $G$  and on the relative sizes of the gauge couplings corresponding to factor groups in the direct product. If the gauge coupling for one of these factor groups  $G_i \subset G$  gets sufficiently strong and dominates over the other(s), then it can produce bilinear fermion condensates that can self-break the  $G_i$  symmetry itself and/or break other gauge symmetries  $G_j \subset G$ .

An example of this dependence of the type of gauge symmetry breaking upon the relative strengths of gauge couplings in a direct-product chiral gauge theory is provided by a modification of the Standard Model (SM) with the same  $N_G = 3$  gauge group  $G_{SM} = \text{SU}(3)_c \otimes \text{SU}(2)_L \otimes \text{U}(1)_Y$  and with the usual fermion content, but with the Higgs field removed. If, at a given scale  $\Lambda_{QCD}$ , the color  $\text{SU}(3)_c$  gauge coupling becomes sufficiently large while the  $\text{SU}(2)_L$  (and  $\text{U}(1)_Y$ ) gauge couplings are weak, then the  $\text{SU}(3)_c$  gauge interaction produces a bilinear quark condensate  $\langle \bar{q}q \rangle$ , which dynamically breaks the electroweak gauge symmetry  $G_{EW} = \text{SU}(2)_L \otimes \text{U}(1)_Y$  to electromagnetic  $\text{U}(1)_{em}$ , giving masses to the  $W$  and  $Z$  bosons. Indeed, this was a motivation for models of dynamical electroweak symmetry breaking by a hypothesized vectorial, asymptotically free gauge interaction that would become strongly coupled at the TeV scale and would produce bilinear fermion condensates involving a set of fermions that are nonsinglets under  $G_{EW}$  [1]. In this scenario, as well as in quantum chromodynamics (QCD) itself, the interaction that becomes strong is vectorial and breaks a weakly coupled chiral gauge interaction to a vectorial subgroup gauge symmetry, namely  $\text{U}(1)_{em}$ . In contrast, as discussed in [2] in the context of the gedanken SM theory with no Higgs field, if the  $\text{SU}(2)_L$  gauge coupling were sufficiently large at a given reference scale, while the  $\text{SU}(3)_c$  gauge coupling were weak, then a very different pattern of symmetry breaking would occur: this  $\text{SU}(2)_L$  gauge interaction would produce bilinear fermion condensates that preserve the  $\text{SU}(2)_L$  gauge invariance but break  $\text{SU}(3)_c$  to  $\text{SU}(2)_c$ , and break  $\text{U}(1)_Y$ , giving masses to the gluons in the coset  $\text{SU}(3)_c/\text{SU}(2)_c$  and to the hypercharge gauge boson.

Chiral gauge theories (without scalars) that are asymptotically free and can therefore become strongly coupled at low energies have been of interest in the past for sev-

eral reasons. One motivation involved an effort to understand the pattern of quark and lepton generations. Since the respective lower bounds on the compositeness scales of these Standard-Model fermions are much larger than their masses, a plausible approach was to begin by using a theoretical framework in which they were massless. Strongly coupled irreducibly chiral gauge theories are a natural candidate for such a framework, since the chiral gauge invariance forbids any fermion mass terms. If such a theory satisfies the 't Hooft global anomaly-matching conditions, then, as the gauge coupling becomes sufficiently strong in the infrared, the gauge interaction could confine and produce massless gauge-singlet composite spin-1/2 fermions [3]-[18].

A different motivation for studying strongly coupled chiral gauge theories arose in the context of models that sought to explain both dynamical electroweak symmetry breaking and fermion mass generation. In terms of low-energy effective Lagrangians, this involved the above-mentioned new vectorial gauge interaction that would become strong at the TeV scale and produce bilinear fermion condensates, in conjunction with a set of four-fermion operators that could give rise to quark and lepton masses [1, 8]. A next step was the construction of ultraviolet-completions of these theories that would have the potential to explain not only the Standard-Model fermion masses in a given generation, but also the existence of a generational hierarchy of fermion masses. A basic property of a chiral gauge theory is that if it becomes strongly coupled, it can produce bilinear fermion condensates that self-break the gauge symmetry [9, 10]. Reasonably UV-complete models for dynamical electroweak symmetry breaking and Standard-Model fermion mass generation made use of this feature (e.g., [12],[13]-[20]). These involved strongly coupled chiral gauge interactions that led to the formation of various fermion condensates which broke the initial chiral gauge symmetry in a sequence of stages that might plausibly explain the SM fermion masses and their generational hierarchy. This sequential breaking was such as to yield, as a residual symmetry, the vectorial gauge symmetry that is strongly coupled at the TeV scale. Ref. [13] used a direct-product chiral gauge group with two strongly coupled gauge interactions and pointed out that different patterns of sequential gauge symmetry breaking (denoted  $G_a$  and  $G_b$  in [13]) could occur, depending on the relative sizes of gauge couplings corresponding to these two factor groups. A similar phenomenon was noted in other models studied in [14]. It is this interesting property of the nonperturbative behavior of direct-product chiral gauge theories that we wish to explore further here.

Another motivation for the present study is the fact that patterns of gauge symmetry breaking by Higgs fields depend on parameters in the Higgs potential  $V$ , which one can choose at will, subject to the constraint that  $V$  should be bounded from below. In contrast, once one has specified the gauge and fermion content of a chiral gauge theory, together with the values of the gauge couplings

at a reference point (which is naturally chosen to be in the deep UV for theories with asymptotically free non-Abelian gauge interactions), then the dynamics determines the pattern of gauge symmetry breaking uniquely [21].

This paper is organized as follows. In Section II we discuss our general theoretical framework, methods of analysis, and a classification of direct-product chiral gauge theories. Section III contains some useful procedures for the construction of (anomaly-free, asymptotically free) chiral gauge theories. In Sections IV-XVI we study a variety of different chiral gauge theories with a direct-product gauge groups and fermion contents. These involve both unitary and orthogonal gauge groups and elucidate how the patterns of dynamical symmetry breaking depend on the structures of the respective theories. Our conclusions are contained in Section XVII.

## II. CLASSIFICATION OF GROUPS AND METHODS OF ANALYSIS

In order to explore the nonperturbative behavior of direct-product chiral gauge theories, it is useful to have a general classification of these theories and general methods for analyzing them. We discuss these in this section. As stated above, we consider direct-product chiral gauge theories with gauge groups of the form (1.1) with fermion content  $\{f\}$  chosen such that the theory is free of any anomalies in gauged currents and free of any global SU(2) Witten anomalies, and also such that all non-Abelian gauge interactions are asymptotically free. Unless otherwise indicated, we will, with no loss of generality, write all fermions as left-handed chiral components.

To describe our classification system, we first introduce some notation. We generically denote a group that has only real or pseudoreal representations as  $G_r$  and a group that has complex representations as  $G_c$ . A group  $G_r$  cannot, by self, be the gauge group of a chiral gauge theory, although it can appear as a factor group in a chiral gauge theory. A group  $G_r$  has zero anomaly, while, in general, a group  $G_c$  has nonzero anomalies  $A_{\mathcal{R}}$  for its representations (see Eq. (A14)), which we will indicate by the symbol  $G_{ca}$ . If a group  $G_c$  has no anomaly, i.e.,  $A_{\mathcal{R}} = 0$  for all  $\mathcal{R}$ , then it is commonly termed “safe” ( $s$ ) [22], and we denote it as  $G_{cs}$ . Of course, a group  $G_r$  is automatically safe. Thus, the generic class  $G_s$  includes  $G_r$  and  $G_{cs}$ .

We may then classify a chiral gauge theory with the direct-product gauge group (1.1) by an  $N_G$ -dimensional vector indicating the nature of the factor groups involved in the direct product. If  $N_G = 1$ , there are two possibilities: (i) ( $ca$ ), e.g., SU( $N$ ) with  $N \geq 3$ , and (ii) ( $cs$ ), e.g., SO( $4k+2$ ) for  $k \geq 2$  or the exceptional group  $E_6$  [22–24]. For  $N_G = 2$ , the possibilities are

$$N_G = 2: \quad (ca, r), (cs, r), (ca, ca), (ca, cs), (cs, cs), \quad (2.1)$$

TABLE I: Classification of some direct-product chiral gauge theories. See text for further discussion.

Type	$N_G$	$G$
$(ca, r), (cav, r)$	2	$SU(N) \otimes SU(2)$ with $N \geq 3$
$(cav, cav)$	2	$SU(N) \otimes SU(M)$ with $N, M \geq 3$
$(r, ca)$	2	$SU(2) \otimes U(1)$
$(ca, ca), (cav, ca)$	2	$SU(N) \otimes U(1)$ with $N \geq 3$
$(cav, r, ca)$	3	$SU(N_c) \otimes SU(2)_L \otimes U(1)_Y$
$(cav, r, r)$	3	$SU(N) \otimes SU(2)_L \otimes SU(2)_R$
$(cav, r, r, cav)$	4	$SU(N_c) \otimes SU(2)_L \otimes SU(2)_R \otimes U(1)_{B-L}$
$(cs, r)$	2	$SO(4k+2) \otimes SU(2)$ with $k \geq 2$
$(cs, cav)$	2	$SO(4k+2) \otimes SU(N)$ with $N \geq 3$
$(cs, cs)$	2	$SO(4k+2) \otimes SO(4k'+2)$ with $k, k' \geq 2$

where we do not distinguish the order of factor groups, so, for example,  $(cs, ca)$  and  $(ca, cs)$  are the same type.

Let us consider a factor group  $G_i$  in (1.1) which is of the form  $G_{ca}$ , and set the gauge couplings of the other factor groups to zero. If the resultant  $G_{ca}$  theory is vectorial ( $v$ ), then we denote this as  $G_{cav}$ . This is the case, for example, with the color  $SU(3)_c$  factor group in the Standard Model. Thus, a further classification of direct-product chiral gauge theories can be carried out in which, for each factor group of the form  $G_{ca}$ , one distinguishes whether or not it is of the form  $G_{cav}$ . The Standard Model gauge group is of the type  $(cav, r, ca)$  in this classification. We illustrate the classification of some chiral gauge theories considered in this paper in Table I.

Our requirement that each non-Abelian factor group in the direct product (1.1) is asymptotically free enables us to describe the theory perturbatively in the deep ultraviolet. We discuss the evolution from the UV to the IR next. To each factor group  $G_i$ ,  $i = 1, \dots, N_G$ , there corresponds a running gauge coupling  $g_i(\mu)$ , and we define  $\alpha_i(\mu) = g_i(\mu)^2/(4\pi)$  and  $a_i(\mu) \equiv g_i(\mu)^2/(16\pi^2)$ . The argument  $\mu$  will often be suppressed in the notation. The UV to IR evolution of the gauge coupling is determined by the beta function,  $\beta_{g_i} = dg_i/dt$ , or equivalently,  $\beta_{G_i} = d\alpha_i/dt = [g/(2\pi)]\beta_{g_i}$ , where  $dt = d \ln \mu$ . This has the series expansion

$$\beta_{G_i} = -8\pi a_i^2 \left[ b_{G_i,1\ell} + \sum_{j=1}^{N_G} b_{G_i,2\ell;ij} a_j \right. \\ \left. + \sum_{j,k=1}^{N_G} b_{G_i,3\ell;ijk} a_j a_k + \dots \right], \quad (2.2)$$

where an overall minus sign is extracted and the dots ... indicate higher-loop terms. Here,  $b_{G_i,1\ell}$  is the one-loop (denoted  $(1\ell)$ ) coefficient, multiplying  $a_i^2$ ,  $b_{G_i,2\ell;ij}$  is the two-loop coefficient, multiplying  $a_i^2 a_j$ , and so forth for higher-loop loops. The property of asymptotic freedom for the non-Abelian gauge interactions means that

$\beta_{G_i} < 0$  for small  $\alpha_i$ ,  $i = 1, \dots, N_{NA}$ . The set (2.2) constitutes a set of  $N_G$  coupled nonlinear first-order ordinary differential equations for the quantities  $\alpha_i$ ,  $i = 1, \dots, N_G$ . To leading order, i.e., to one-loop order, the set of differential equations decouple from each, and one has the simple solution for each  $i \in \{1, \dots, N_G\}$ :

$$\alpha_i(\mu_1)^{-1} = \alpha_i(\mu_2)^{-1} - \frac{b_{G_i,1\ell}}{2\pi} \ln \left( \frac{\mu_2}{\mu_1} \right), \quad (2.3)$$

where we take  $\mu_1 < \mu_2$ .

In the following discussion, we assume that the fundamental Lagrangian has no fermion mass terms, so that all fermion masses are generated dynamically by chiral symmetry breaking. For a pair of gauge interactions corresponding to the factor groups  $G_i$  and  $G_j$  in Eq. (1.1), the respective beta functions  $\beta_{G_i}$  and  $\beta_{G_j}$  in the deep UV are fixed once we choose the fermion content of a given theory. The values of the corresponding  $\alpha_i(\mu_1)$  and  $\alpha_j(\mu_1)$  at lower Euclidean scales are determined by (i) the initial values of  $\alpha_i(\mu_2)$  and  $\alpha_j(\mu_2)$  in the UV; (ii) the values of  $\beta_{G_i}$  and  $\beta_{G_j}$ ; and (iii) the occurrence of bilinear fermion condensate formation at some scale(s) as the theory evolves from the deep UV toward the IR, which produce dynamical masses for the fermions involved in these condensates. Since we do not assume that the direct-product group (1.1) is contained in a simple group in the deep UV, we are free to consider various different orderings of the sizes of the couplings  $\alpha_i(\mu_2)$  in the UV. Furthermore, because of the condensation process(es) (iii), the fermions involved in these condensates, together with gauge bosons corresponding to broken generators of gauge symmetries, acquire dynamical masses and are integrated out of the low-energy effective field theories that are applicable as the Euclidean reference scale decreases below each condensation scale. The reduction in massless particle content in (iii) produces changes in the beta functions of the gauge interactions involved. Because of this, even if  $\beta_{G_i} > \beta_{G_j}$  with all fermions initially present in the deep UV, it can happen that at a lower scale this inequality is reversed. The variation of gauge couplings in the deep UV embodied in the input (i) above was carried out in the earlier work [13] where both of the cases of relative sizes of  $\alpha_{ETC}$  and  $\alpha_{HC}$  were considered, and in [2], where both of the cases of relative sizes of couplings for  $SU(3)_c$  and  $SU(2)_L$  were considered. Henceforth, for notational simplicity, we set  $b_{G_i,1\ell} \equiv b_{G_i,1}$ . There have been a number of interesting studies of renormalization-group (RG) flows in quantum field theories with multiple interaction couplings using perturbatively calculated beta functions, e.g., [25]. Here, as in the earlier works involving gauge theories with multiple gauge couplings [2, 13, 15], we will focus on the nonperturbative phenomenon of fermion condensate formation and the associated pattern of gauge symmetry breaking. The one-loop result (2.3) will be sufficient for our purposes here since we focus on this nonperturbative fermion condensate formation. These condensates also generically break global chiral symmetries.

In general, a fermion condensate may involve different fermion fields or the same fermion field. If the fields are the same, we may write the bilinear fermion operator product abstractly as follows. Assume that the gauge group  $G$  in Eq. (1.1) contains  $t \leq N_G$  non-Abelian factors  $G_k$  and that the relevant fermion field  $f$  transforms as the representation  $\mathcal{R} \equiv (\mathcal{R}_1, \dots, \mathcal{R}_t)$  under the direct product of these non-Abelian factor groups. Then the bilinear fermion product of a given fermion field is

$$f_{\mathcal{R},i,L}^T C f_{\mathcal{R},j,L}, \quad (2.4)$$

where  $C$  is the Dirac conjugation matrix, gauge group indices are suppressed in the notation, and  $i, j$  are copy (flavor) indices. From the property  $C^T = -C$  together with the anticommutativity of fermion fields, it follows that the bilinear fermion operator product (2.4) is symmetric under interchange of the order of fermion fields and therefore is symmetric in the overall product

$$\left[ \prod_{k=1}^t (\mathcal{R}_k \times \mathcal{R}_k) \right] S_{ij}, \quad (2.5)$$

where  $S_{ij}$  abstractly denotes the symmetry property under interchange of flavors, with  $S_{ij} = (ij)$  and  $S_{ij} = [ij]$  for symmetric and antisymmetric flavor structure, respectively. For example, for the case  $t = N_g = 2$  and flavor indices  $i, j$ , the symmetry property (2.5) means that  $f_{i,L}^T C f_{j,L}$  is of the form  $(s, s, s)$ ,  $(s, a, a)$ ,  $(a, s, a)$ , or  $(a, a, s)$ , where here  $s$  and  $a$  indicate symmetric and antisymmetric and the three entries refer to the representations  $\mathcal{R}_1$  of  $G_1$ ,  $\mathcal{R}_2$  of  $G_2$ , and  $S_{ij}$ . Thus, as an illustration, in the last case,  $(a, a, s)$ , the product (2.4) would transform as antisymmetric representations in the Clebsch-Gordan products of  $\mathcal{R}_j \times \mathcal{R}_j$  for  $j = 1, 2$  and would be symmetric in flavor indices, with  $S_{ij} = (ij)$ , and so forth for other cases.

The main perturbative information that we will use is the one-loop coefficients of the beta functions for the non-Abelian gauge interactions. We require that these interactions must be asymptotically free so that we have perturbative control over them in the deep UV. If  $\alpha_i(\mu)$  becomes strong, i.e.,  $O(1)$  in the IR, one can no longer use perturbative methods reliably, but one can make use of several approximate methods to explore possible non-perturbative properties of the theory. First, one may investigate whether the fermions in the theory satisfy the 't Hooft anomaly-matching conditions. For this purpose, one determines the global flavor symmetry group of the theory is invariant and then checks whether candidate operators for gauge-singlet composite spin-1/2 fermions match the anomalies in the global flavor symmetries. If this necessary condition is satisfied, then it is possible that in the infrared the strong chiral gauge interaction could confine and produce massless composite spin-1/2 fermions.

A different possibility in a strongly coupled chiral gauge theory is that the gauge interaction can produce

bilinear fermion condensates. This will be the main focus of our analysis here. In an irreducibly chiral theory these condensates break one or more gauge symmetries, as well as global flavor symmetries. A commonly used method for suggesting which type of condensate is most likely to form in this case is the most-attractive-channel (MAC) method [10]. For possible condensation of chiral fermions in the representations  $\mathcal{R}_{G_i,1}$  and  $\mathcal{R}_{G_i,2}$  of the factor group  $G_i$  in (1.1) in various channels of the form  $\mathcal{R}_{G_i,1} \times \mathcal{R}_{G_i,2} \rightarrow \mathcal{R}_{G_i,cond.}$ , the MAC approach predicts that the condensation will occur in the channel with the largest (positive) value of the quantity

$$\Delta C_2 \equiv C_2(\mathcal{R}_{G_i,1}) + C_2(\mathcal{R}_{G_i,2}) - C_2(\mathcal{R}_{G_i,cond.}), \quad (2.6)$$

where  $C_2(\mathcal{R})$  is the quadratic Casimir invariant for the representation  $\mathcal{R}$  (see Appendix A). This is only a rough measure, based on one-gluon exchange. The form of the condensate determines the resultant symmetry and form of vacuum alignment [11].

### III. METHODS FOR CONSTRUCTING CHIRAL GAUGE THEORIES

In this section we mention some useful methods for constructing anomaly-free direct-product chiral gauge theories.

#### A. Reduction Method

Let us say that we have a chiral gauge theory with the  $N_G$ -fold direct product gauge group (1.1) and a given fermion content that satisfies the constraints that the theory must be free of any anomaly in gauged currents, any possible global  $SU(2)$  anomaly, and, if  $G$  includes abelian factor groups, also any mixed gravitational-gauge anomaly. One can then construct a set of chiral gauge theories by a process of reduction, setting one or more of the gauge couplings  $\{g_1, \dots, g_{N_G}\}$  equal to zero. As an example, if one starts with a modified and extended Standard Model with gauge group (7.1) and fermion content (7.2)-(7.4) below, of type  $(cav, r, ca)$ , then (i) by turning off the  $SU(N_c)$  gauge coupling, one gets an  $SU(2)_L \otimes U(1)_Y$  gauge theory of type  $(r, ca)$ ; (ii) by turning off the  $SU(2)_L$  gauge coupling, one gets an  $SU(N_c) \otimes U(1)_Y$  gauge theory of type  $(cav, ca)$ ; and (iii) by turning off the  $U(1)_Y$  coupling, one gets an  $SU(N_c) \otimes SU(2)_L$  gauge theory of type  $(cav, r)$ . Given that the original theory has the requisite property that all non-Abelian gauge interactions are asymptotically free, the theory derived by turning off some gauge coupling(s) also has this property.

### B. Extension Method to Construct $G = \tilde{G} \otimes G_s$ Theories

Here we present a method for constructing a direct-product chiral gauge theory with an  $(N_G + 1)$ -fold direct-product gauge group, starting from a given chiral gauge theory with an  $N_G$ -fold direct-product gauge group  $\tilde{G}$  by adjoining a safe group  $G_s$  to  $\tilde{G}$  to produce

$$G = \tilde{G} \otimes G_s \quad (3.1)$$

and extending the fermion representations of  $\tilde{G}$  to those of  $G = \tilde{G} \otimes G_s$ . Here  $G_s$  may be  $G_r$  or  $G_{cs}$ . The procedure is as follows:

1. Start with an anomaly-free chiral gauge theory with the  $N_G$ -fold gauge group  $\tilde{G} = \bigotimes_{i=1}^{N_G} G_i$  and a set of fermion representations  $\{\mathcal{R}_{\tilde{G}}\}$ , where each of these is

$$\mathcal{R}_{\tilde{G}} = (\mathcal{R}_{G_1}, \dots, \mathcal{R}_{G_{N_G}}) \quad (3.2)$$

2. Choose the safe group  $G_s$ , of type  $G_r$  or  $G_{cs}$ , i.e., either a group with real representations, such as  $SU(2)$ , or a safe group with complex representations, such as  $SO(4k+2)$  with  $k \geq 2$  or the exceptional group  $E_6$ .
3. Extend each fermion representation  $\mathcal{R}_{\tilde{G}}$  of  $\tilde{G}$  to a representation  $\mathcal{R}_G$  of  $G$  using a single representation  $\mathcal{R}_{G_s}$  of  $G_s$  to form  $\mathcal{R}_G = (\mathcal{R}_{\tilde{G}}, \mathcal{R}_{G_s})$ . As far as the  $\tilde{G}$  group is concerned, this simply amounts to a replication of its original (anomaly-free) fermion content by  $\dim(\mathcal{R}_{G_s})$  copies, so the resulting extended fermion content is also anomaly-free.
4. Apply the constraint that if the safe group is  $G_s = SU(2)$ , then the resultant theory must be free of a global  $SU(2)$  Witten anomaly associated with the homotopy group  $\pi_4(SU(2)) = \mathbb{Z}_2$  [26, 27]. With  $\mathcal{R}_{G_s} = \square$ , the necessary and sufficient condition to satisfy this constraint is that the total number of  $SU(2)$  doublets is even [26].
5. Apply the constraints that each of the gauge interactions corresponding to non-Abelian factor groups in  $\tilde{G}$  must remain asymptotically free in the larger group  $G$ , and the  $G_s$  gauge interaction must also be asymptotically free.

This method can be used to construct many types of direct-product chiral gauge groups. Among the  $N_G = 2$  cases, for example, these types include all of the ones listed in Eq. (2.1).

### IV. $G_{cav} \otimes SU(2)$ THEORIES

In this section we construct and study a class of  $N_G = 2$  direct-product chiral gauge theories with a gauge group

$$G_1 \otimes G_2 = G_{cav} \otimes SU(2) . \quad (4.1)$$

This class is the special case  $(cav, r)$  of the class  $G_{ca} \otimes G_r$  discussed in Section II in which  $G_{ca} = G_{cav}$ , i.e.,  $G_{ca}$  is a group with complex representations and  $\mathcal{A}_{\mathcal{R}} \neq 0$  and the fermion content is such that if the  $SU(2)$  gauge interaction is turned off, then the  $G_{cav}$  gauge interaction is vectorial. This property guarantees that there is no cubic triangle anomaly in gauged currents in the  $G_{cav}$  sector. Furthermore, as already indicated above, since  $SU(2)$  has (pseudo)real representations, it has no anomaly. The only anomaly constraint is then the requirement that the  $SU(2)$  group must be free of a global anomaly. We consider theories of this type with chiral fermion content (written here as left-handed)

$$\{f_{ns,ns}\} = \sum_{\mathcal{R}} p_{\mathcal{R}} (\mathcal{R}, \square) , \quad (4.2)$$

$$\{f_{ns,s}\} = 2 \sum_{\mathcal{R}} p_{\mathcal{R}} (\bar{\mathcal{R}}, 1) , \quad (4.3)$$

and optionally,

$$\{f_{s,ns}\} = p_1 (1, \square) , \quad (4.4)$$

where the subscripts  $ns$  and  $s$  are abbreviations for “non-singlet” and “singlet”;  $\mathcal{R}$  denotes a (nonsinglet) representation of the group  $G_1$ ; and the first and second entries in the subscripts and in the parentheses refer to the representations of  $G_{cav}$  and  $SU(2)_L$ , respectively, with  $\square$  being the fundamental representation in standard Young tableaux notation.

If the fermion sector includes only a single  $\mathcal{R}$ , then we set  $p_{\mathcal{R}} \equiv p$  for brevity. We shall use interchangeably a notation with Young tableaux and dimensionalities to identify the representation:  $(\mathcal{R}, \square) \leftrightarrow (\dim(\mathcal{R}), 2)$ . In general, we will allow for several types of (nonsinglet) representations  $\mathcal{R}$ , but will focus on minimal theories with only one  $\mathcal{R}$ . The subscript indices  $i, j$  are copy (“flavor”) indices, and the total number of copies of the  $f_{ns,ns}$  fermions transforming as the  $\mathcal{R}$  representation of  $G_1$  is denoted  $p_{\mathcal{R}}$ . We shall mainly focus on irreducibly chiral theories, i.e., those for which the chiral gauge theory forbids any bare mass terms, but we shall also discuss some chiral gauge theories with vectorlike subsectors. The global symmetries depend on  $p$  and  $p_1$ ; we will discuss them for specific models below.

The number of  $SU(2)$  chiral fermion doublets in this theory, which we shall denote  $N_d$ , is

$$N_d = p_1 + \sum_{\mathcal{R}} p_{\mathcal{R}} \dim(\mathcal{R}) . \quad (4.5)$$

The condition that the  $SU(2)$  gauge sector must be free of a global anomaly is that

$$N_d \text{ is even} . \quad (4.6)$$

Because  $N_d$  is necessarily even, one could take half of the left-handed  $SU(2)$ -doublet fermions, rewrite them

as right-handed charge-conjugates, and thereby put the SU(2) gauge interaction into vectorial form.

As noted, we shall also impose two further requirements on the theory, namely that the  $G_1$  and the SU(2) gauge interactions must both be asymptotically free. From the general results in [28], we find that the one-loop coefficient of the beta function of the  $G_1$  gauge interaction is

$$b_{1,G_1} = \frac{1}{3} \left[ 11C_2(G_1) - 8 \sum_{\mathcal{R}} p_{\mathcal{R}} T(\mathcal{R}) \right], \quad (4.7)$$

so the requirement that the  $G_1$  gauge interaction should be asymptotically free implies that

$$\sum_{\mathcal{R}} p_{\mathcal{R}} T(\mathcal{R}) < \frac{11C_2(G_1)}{8}. \quad (4.8)$$

Here and below, if  $p_1 = 0$  and the theory contains fermions in one (nonsinglet) representation  $\mathcal{R}$  of  $G_1$ , then only nonzero values of  $p_{\mathcal{R}} \equiv p$  are relevant, since if  $p = 0$ , then the theory is a pure (direct-product) gauge theory and hence is not a chiral gauge theory.

The one-loop coefficient of the beta function of the SU(2) gauge interaction is

$$\begin{aligned} b_{1,\text{SU}(2)_L} &= \frac{1}{3}(22 - N_d) \\ &= \frac{1}{3} \left[ 22 - \left( p_1 + \sum_{\mathcal{R}} p_{\mathcal{R}} \dim(\mathcal{R}) \right) \right], \end{aligned} \quad (4.9)$$

so the requirement that the SU(2) gauge interaction should be asymptotically free implies that

$$p_1 + \sum_{\mathcal{R}} p_{\mathcal{R}} \dim(\mathcal{R}) < 22. \quad (4.10)$$

## V. SU(N) $\otimes$ SU(2) THEORIES

In this section we construct and study several models with a direct-product gauge group of the form (4.1) with the first gauge group being SU( $N$ ), i.e., with

$$G = G_1 \otimes G_2 = \text{SU}(N) \otimes \text{SU}(2) \quad (5.1)$$

and various chiral fermion contents, which we denote as Models A, B, and C. All three of these models are of type (*cav*,  $r$ ), as indicated in Table I.

### A. Model A

The first model that we consider, denoted Model A, is a minimal one in three respects: (i) it contains no  $G_1$ -singlet fermions, i.e.,  $p_1 = 0$ ; (ii) the fermions transform according to only one representation  $\mathcal{R}$  of  $G_1$  and its conjugate; and (iii) this representation  $\mathcal{R}$  is the simplest

nontrivial one, namely the fundamental,  $\mathcal{R} = \square$ . The chiral fermions are

$$\psi_{i,L}^{a,\alpha}, \quad i = 1, \dots, p : \quad p(\square, \square) = p(N, 2), \quad (5.2)$$

and

$$\chi_{a,j,L}, \quad j = 1, \dots, 2p : \quad 2p(\bar{\square}, 1) = 2p(\bar{N}, 1). \quad (5.3)$$

Here,  $a$  and  $\alpha$  are SU( $N$ ) and SU(2) gauge indices and  $i, j$  are copy (“flavor”) indices. For  $N \geq 3$ , the chiral gauge symmetry forbids any bare mass terms for the fermions. In contrast, if  $N = 2$ , then gauge-invariant bare mass terms such as

$$\epsilon^{ab} \chi_{a,i,L}^T C \chi_{b,j,L}, \quad i \neq j, \quad 1 \leq i, j \leq 2p \quad (5.4)$$

and

$$\epsilon_{ab} \epsilon_{\alpha\beta} \psi_{i,L}^{a,\alpha T} C \psi_{j,L}^{b,\beta}, \quad 1 \leq i, j \leq p \quad (5.5)$$

can occur. Closely related to this, if  $N = 2$ , then the SU( $N$ ) and SU(2) gauge interactions can both be written in vectorial form, so the theory is not a chiral gauge theory. Therefore, henceforth we shall assume that  $N \geq 3$  for this class of theories. In the notation introduced above, the fermion content of this Model A can be categorized as being of the form

$$\{f_{ns,ns}, f_{ns,s}\}. \quad (5.6)$$

The fermion terms in the Lagrangian for this model are

$$\mathcal{L} = \sum_{j=1}^p \bar{\psi}_{j,L} i \not{D} \psi_{j,L} + \sum_{j=1}^{2p} \bar{\chi}_{j,L} i \not{D} \chi_{j,L}, \quad (5.7)$$

(where we have indicated the sums over flavor indices explicitly). In connection with the discussions in Sections III A and VII, we note that one realization of a Model A theory is the gauge and quark sector of the generalized Standard Model with the Higgs field removed, the weak hypercharge gauge coupling turned off, and with the identifications  $N = N_c$  and  $p = N_g$ , where  $N_g$  denotes the number of fermion generations. In this case, the correspondence of fermion fields here and in Eqs. (7.2) and (7.3) is as given below in Eqs. (7.11) and (7.12). This correspondence motivates the property that the Lagrangian (5.7) is diagonal in copy indices; if one were to include terms of the form  $\bar{\chi}_{j,L} i \not{D} \chi_{k,L}$  with  $j \neq k$ , some of these would correspond, in the generalized SM, to terms of the form  $\bar{u}_{j',L} i \not{D} d_{k',L}$  that would violate  $U(1)_Y$  and electromagnetic  $U(1)_{em}$  gauge symmetries. Although Model A has no  $U(1)_Y$  factor, we will restrict the Lagrangian to the form (5.7) which could be derived from the generalized SM by the reduction process of Section III A.

For this Model A, the condition that the SU(2) gauge sector should be free of a global anomaly is

$$N_d = pN \text{ is even}, \quad (5.8)$$

TABLE II: Values of  $N$  and  $p$  in the Model A  $SU(N) \otimes SU(2)_L$  chiral gauge theory allowed by the inequalities (5.10) and (5.12) arising from the constraint of asymptotic freedom for the  $SU(N)$  and  $SU(2)$  gauge interactions, respectively, and the requirement that the theory must not have any global  $SU(2)$  anomaly, Eq. (5.8). The notation  $12 \leq N_{\text{even}} \leq 20$  denotes the even values of  $N$  in this range. The notation  $13 \leq N_{\text{odd}} \leq 21$  denotes the odd values of  $N$  in this range. For  $N \geq 22$ , the inequality (5.12) has only the trivial solution  $p = 0$  for which the theory is a pure gauge theory with no fermions and hence is not a chiral gauge theory.

$N$	allowed values of $p$
3	$p = 2, 4, 6$
4	$1 \leq p \leq 5$
5	$p = 2, 4$
6	$1 \leq p \leq 3$
7	$p = 2$
8	$p = 1, 2$
9	$p = 2$
10	$p = 1, 2$
11	no sol. with $p \neq 0$
$12 \leq N_{\text{even}} \leq 20$	$p = 1$
$13 \leq N_{\text{odd}} \leq 21$	no sol. with $p \neq 0$
$N \geq 22$	no sol. with $p \neq 0$

and we require that this condition must be satisfied.

From the general result (4.7), we have, for the one-loop coefficient of the  $SU(N)$  beta function,

$$b_{1,SU(N)} = \frac{1}{3}(11N - 4p). \quad (5.9)$$

Therefore, the requirement that the  $SU(N)$  gauge interaction should be asymptotically free, expressed by the inequality (4.8), reads

$$p < \frac{11N}{4}. \quad (5.10)$$

From the general result (4.9), we find, for the one-loop coefficient of the  $SU(2)$  beta function,

$$b_{1,SU(2)_L} = \frac{1}{3}(22 - pN). \quad (5.11)$$

Hence, the requirement that the  $SU(2)$  gauge interaction should be asymptotically free, given by the inequality (4.10), is

$$pN < 22. \quad (5.12)$$

In Fig. 1 we show the boundaries of the region in the  $(N, p)$  plane satisfying the inequalities (5.10) and (5.12). The allowed values of  $N$  and  $p$  are thus the integers  $N \geq 3$  and  $p \geq 1$  in this allowed region that satisfy the conditions (5.10), (5.12), and (5.8). We list these in Table II.

Several comments are in order concerning these allowed values of  $N$  and  $p$ . First, as  $N$  increases through the

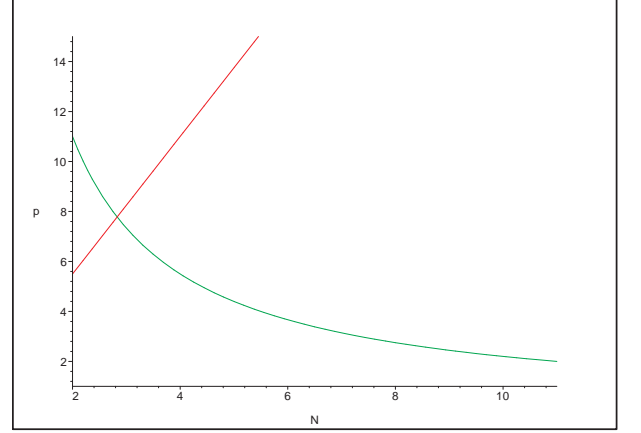


FIG. 1: Plot of the region in  $N$  and  $p$  allowed by the requirement of asymptotic freedom for the  $SU(N)$  and  $SU(2)$  gauge interactions in the  $SU(N) \otimes SU(2)_L$  Model A chiral gauge theory. The boundaries of this region are given by the line from the inequality (5.10) and the hyperbola from the inequality (5.12). The allowed values of  $N$  and  $p$  are thus the integers  $N \geq 3$  and  $p \geq 1$  in this allowed region that also satisfy the condition that the theory must not have any global  $SU(2)$  anomaly, Eq. (5.8). See text for further discussion.

value  $N = 22$ , the maximum value of  $p$  allowed by the inequality (5.12) decreases below 1, so that for  $N > 22$ , this inequality (5.12) has only the trivial (integral) solution  $p = 0$  for which the theory is a pure gauge theory with no fermions and hence not of interest here. Second, for odd  $N$ , one sees that the condition (5.8) for the theory to be free from a global  $SU(2)$  anomaly restricts  $p$  to even values.

We next analyze the UV to IR evolution and gauge symmetry breaking in this model. If the  $SU(N)$  gauge interaction is sufficiently strong and if it dominates over the  $SU(2)$  gauge interaction, then this  $SU(N)$  interaction forms bilinear fermion condensates that break the  $SU(2)$  gauge symmetry. We denote the scale at which this occurs as  $\Lambda$ . As regards the  $SU(N)$  gauge interaction, the most attractive channel for fermion condensation is

$$SU(N) : \quad \square \times \bar{\square} \rightarrow 1, \quad (5.13)$$

in terms of Young tableaux, or equivalently,  $N \times \bar{N} \rightarrow 1$ , in terms of the dimensionalities of the  $SU(N)$  representations, with associated condensates

$$\left\langle \sum_{a=1}^N \psi_{i,L}^{a,\alpha} {}^T C \chi_{a,j,L} \right\rangle, \quad (5.14)$$

where  $i \in \{1, \dots, p\}$  and  $j \in \{1, \dots, 2p\}$ . (Here and below, when a condensate is given, it is understood that the

hermitian conjugate condensate is also present.) This channel has

$$\Delta C_2 = 2C_2(\square) = \frac{N^2 - 1}{N}. \quad (5.15)$$

Each of the condensates in Eq. (5.14) breaks the SU(2) gauge symmetry completely (and is invariant under the SU(N) gauge symmetry, as is clear from (5.13)). The fermions involved in these condensates, and the SU(2) gauge bosons, gain dynamical masses of order  $\Lambda$ .

If, on the other hand, the SU(2) interaction is sufficiently strong and if it dominates over the SU(N) interaction, then this SU(2) interaction produces bilinear fermion condensates in the most attractive SU(2) channel  $2 \times 2 \rightarrow 1$ , with associated condensates of the form

$$\langle \epsilon_{\alpha\beta} \psi_{i,L}^{a,\alpha} {}^T C \psi_{j,L}^{b,\beta} \rangle. \quad (5.16)$$

We denote the scale where this occurs as  $\Lambda'$ . The attractiveness measure for condensate formation in this channel is  $\Delta C_2 = 2C_2(\square) = 3/2$ . From the general symmetry property (2.5), it follows that if, as in Eq. (5.16), one contracts the SU(2) gauge indices  $\alpha$  and  $\beta$  antisymmetrically via the SU(2)  $\epsilon_{\alpha\beta}$  tensor, then the combination of SU(N) and generational indices is antisymmetric. That is, in the operator product (5.16), either the SU(N) gauge indices are antisymmetric and the generational indices are symmetric, so the condensate is proportional to

$$\begin{aligned} & \langle \epsilon_{\alpha\beta} (\psi_{i,L}^{a,\alpha} {}^T C \psi_{j,L}^{b,\beta} - \psi_{i,L}^{b,\alpha} {}^T C \psi_{j,L}^{a,\beta} \\ & + \psi_{j,L}^{a,\alpha} {}^T C \psi_{i,L}^{b,\beta} - \psi_{j,L}^{b,\alpha} {}^T C \psi_{i,L}^{a,\beta}) \rangle \end{aligned} \quad (5.17)$$

or the SU(N) gauge indices are symmetric and the generational indices are antisymmetric, so the condensate is proportional to

$$\begin{aligned} & \langle \epsilon_{\alpha\beta} (\psi_{i,L}^{a,\alpha} {}^T C \psi_{j,L}^{b,\beta} + \psi_{i,L}^{b,\alpha} {}^T C \psi_{j,L}^{a,\beta} \\ & - \psi_{j,L}^{a,\alpha} {}^T C \psi_{i,L}^{b,\beta} - \psi_{j,L}^{b,\alpha} {}^T C \psi_{i,L}^{a,\beta}) \rangle. \end{aligned} \quad (5.18)$$

The SU(N) gauge interaction, although assumed to be weaker than the SU(2) gauge interaction, is not assumed to be negligible, and it prefers the condensation channel that is the MAC as regards SU(N). Now

$$\Delta C_2 = \frac{N+1}{N} \quad \text{for} \quad \square \times \square \rightarrow \square \quad (5.19)$$

whereas

$$\Delta C_2 = -\frac{N-1}{N} \quad \text{for} \quad \square \times \square \rightarrow \square\square \quad (5.20)$$

so the  $\square \times \square \rightarrow \square$  channel is the MAC, and indeed, the  $\square \times \square \rightarrow \square\square$  channel is repulsive. Therefore, we conclude that in this case where SU(2) is more strongly coupled than SU(N), the expected condensation channel is, in an obvious notation,

$$(\square, 2) \times (\square, 2) \rightarrow (\square, 1) \quad (5.21)$$

with associated condensate (5.17). This condensate, which is of the form  $\langle T^{[ab]} \rangle$ , where  $T^{[ab]}$  is a rank-2 antisymmetric tensor of SU(N), breaks SU(N) as follows [29]:

$$\langle T^{[ab]} \rangle : \text{SU}(N) \rightarrow H = \begin{cases} \text{SU}(2) & \text{if } N = 3 \\ \text{SU}(N-2) \otimes \text{SU}(2) & \text{if } N \geq 4 \end{cases} \quad (5.22)$$

The fermions involved in the condensate and the gauge bosons in the coset SU(N)/H gain dynamical masses of order  $\Lambda'$  and are integrated out of the low-energy effective field theory that is operative as the reference scale  $\mu$  decreases below  $\Lambda'$ . The fermion condensates that form in both the strong-SU(N) and strong-SU(2) situations also break global flavor symmetries. Since we have already analyzed this sort of global flavor symmetry breaking in our previous works [17, 18], we will not pursue this here, instead focusing on the gauge symmetry breaking.

## B. Model B

This model, denoted Model B, has the same gauge group as Model A, but has an enlarged chiral fermion sector which also contains  $p_1 \equiv p'$  copies of the SU(N)-singlet, SU(2)-doublet fermion

$$\eta_{j,L}^\alpha, \quad j = 1, \dots, p' : \quad p'(1, 2). \quad (5.23)$$

Thus, the fermion content of Model B can be categorized as being of the form

$$\{f_{ns,ns}, f_{ns,s}, f_{s,ns}\} \quad (5.24)$$

in the notation of Eq. (5.6). Depending on the value of  $p'$ , these additional fermions may have gauge-invariant bare mass terms of the form

$$\epsilon_{\alpha\beta} \eta_{i,L}^\alpha {}^T C \eta_{j,L}^\beta, \quad (5.25)$$

where  $i \neq j$  and  $1 \leq i, j \leq p'$ . Using the general symmetry property (2.5) and taking account of the antisymmetric contraction of the SU(2) gauge indices  $\alpha$  and  $\beta$  with the  $\epsilon_{\alpha\beta}$  tensor, it follows that the fermion operator in (5.25) is automatically antisymmetrized in the flavor indices  $i$  and  $j$ , so if  $p' = 1$ , then it vanishes identically. If  $p' \geq 2$ , then the  $\{f_{s,ns}\}$  fermions constitute a vectorlike subsector in the full chiral gauge theory.

The sector of SU(N)-nonsinglet fields in Model B is the same as in Model A, so the SU(N) gauge interaction is again vectorial and hence is free from any gauge anomaly, as is the SU(2) gauge interaction. The condition that the SU(2) part of the theory should be free of any global anomaly is that the number of SU(2) doublets, denoted  $N_d$ , is even, i.e.,

$$N_d = pN + p' \text{ is even}, \quad (5.26)$$

and we require that this condition be satisfied.



The one-loop coefficient of the  $SU(N)$  beta function,  $b_{1,SU(N)}$ , is given by (5.9), as in Model A, so  $p$  is subject to the same upper bound from the requirement that the  $SU(N)$  interaction must be asymptotically free, namely (5.10).

The one-loop coefficient of the  $SU(2)$  beta function is

$$b_{1,SU(2)_L} = \frac{1}{3}[22 - (pN + p')] , \quad (5.27)$$

so the requirement that the  $SU(2)$  gauge interaction should be asymptotically free implies that

$$pN + p' < 22 . \quad (5.28)$$

The allowed values of  $N$ ,  $p$ , and  $p'$  for Model B are thus the integers  $N \geq 3$ ,  $p \geq 1$ , and  $p' \geq 1$  satisfying the conditions (5.10), (5.28), and (5.26). There are too many values to list in a table analogous to Table II, but we mention that for  $N = 3$ , the allowed values of  $(p, p')$  are  $(1, 2k + 1)$  with  $0 \leq k \leq 8$ ;  $(2, 2k)$  with  $1 \leq k \leq 7$ ;  $(3, 2k + 1)$  with  $0 \leq k \leq 5$ ;  $(4, 2k)$  with  $1 \leq k \leq 4$ ;  $(5, 2k + 1)$  with  $0 \leq k \leq 2$ ; and the single pair  $(6, 2)$ . As in Model A, as  $N$  increases, the allowed set of values of  $p$  and  $p'$  is progressively reduced, and for sufficiently large  $N$ , there are no nontrivial solutions to the three conditions. For example, for  $N = 16$ , there are only two allowed sets of  $(p, p')$ , namely  $(1, 2)$  and  $(1, 4)$ ; for  $N = 17$ , there are again two sets, namely  $(1, 1)$  and  $(1, 3)$ , while for  $N = 18$ , there is only one,  $(1, 2)$ , and for  $N = 19$ , there is only one,  $(1, 1)$ . For  $N \geq 20$ , there are no allowed (nonzero) values of  $p$  and  $p'$  in this model.

Since Model B is the same as Model A as regards the  $SU(N)$ -nonsinglet fermion content, it follows that if the  $SU(N)$  gauge interaction is sufficiently strong and dominates over the  $SU(2)$  interaction, then the resultant bilinear fermion condensate formation is the same as in Model A.

However, if the opposite is the case, i.e., if the  $SU(2)$  gauge interaction is strong enough and dominates over the  $SU(N)$  interaction, then, depending on the value of  $p'$ , two additional type of fermion condensates may be produced. These all have the same  $SU(2)$  attractiveness measure, as given before, namely,  $\Delta C_2 = 3/2$  and hence, if  $SU(N)$  interactions are negligible, they are expected to form at essentially the same Euclidean scale, which we again denote as  $\Lambda$ . Thus, in addition to the condensate(s) (5.17), the  $SU(2)$  gauge interaction can lead to condensation in the channel

$$(1, 2) \times (1, 2) \rightarrow (1, 1) \quad (5.29)$$

with the associated condensate(s)

$$\langle \epsilon_{\alpha\beta} \eta_{i,L}^{\alpha T} C \eta_{j,L}^{\beta} \rangle , \quad (5.30)$$

where  $1 \leq i, j \leq p'$ . From (2.5), it follows that the bilinear fermion operator product in (5.30) is antisymmetric in the copy indices  $i$  and  $j$  and hence vanishes identically if  $p' = 1$ . As is evident from (5.29), this condensate (5.30)

preserves the full  $SU(N) \otimes SU(2)_L$  gauge symmetry. The fermions involved in these condensates gain dynamical masses of order the condensation scale, denoted  $\Lambda$ , and are integrated out in the low-energy effective field theory that is operative as the reference scale  $\mu$  decreases below  $\Lambda$ .

The second possible additional condensation channel is

$$(N, 2) \times (1, 2) \rightarrow (N, 1) \quad (5.31)$$

with the associated condensate(s)

$$\langle \epsilon_{\alpha\beta} \psi_{i,L}^{a,\alpha T} C \eta_{j,L}^{\beta} \rangle , \quad (5.32)$$

where  $1 \leq i \leq p$  and  $1 \leq j \leq p'$ . Consider the condensates (5.32) with a given  $i$ , say  $i = 1$ . This set of condensates (5.32) breaks  $SU(N)$  to  $SU(N - p')$  if  $1 \leq p' \leq N - 2$  and breaks  $SU(N)$  completely if  $p' \geq N - 1$ . To show this, note that without loss of generality we may pick  $a = N$  and  $j = 1$  for one of these condensates. This condensate,  $\langle \epsilon_{\alpha\beta} \psi_{1,L}^{N,\alpha T} C \eta_{1,L}^{\beta} \rangle$ , breaks  $SU(N)$  to the subgroup  $SU(N - 1)$ . The fermions  $\psi_{1,L}^{N,\alpha}$  and  $\eta_{1,L}^{\beta}$  involved in this condensate gain dynamical masses of order the scale at which this condensate forms. Next, consider the condensate of the form (5.32), where now only the  $SU(N - 1)$  gauge indices  $a \in \{1, \dots, N - 1\}$  are dynamical. Again, by convention, we may pick the  $SU(N - 1)$  gauge index in this condensate to be  $N - 1$  and the copy index on the  $\eta_{j,L}^{\beta}$  fermion to be  $j = 2$ . This breaks  $SU(N - 1)$  to  $SU(N - 2)$  and the fermions  $\psi_{1,L}^{N-1,\alpha}$  and  $\eta_{2,L}^{\beta}$  involved in this condensate gain dynamical masses of order the condensation scale. This process continues until  $SU(N)$  is broken to  $SU(N - p')$  if  $N - p' \geq 2$  or until  $SU(N)$  is completely broken if  $N - p' \leq 1$ . A vacuum alignment argument suggest that it is plausible that this pattern of breaking would also hold for other values  $i = 2, \dots, p$ . As noted above, since the  $SU(2)$  attractiveness measure of all of these condensates,  $\Delta C_2 = 3/2$  is the same, one expects that they form at essentially the same scale.

## VI. EXAMINATION OF SOME OTHER $SU(N) \otimes SU(2)$ THEORIES

Here we examine some  $N_G = 2$  chiral gauge theories with gauge groups of the form  $G_1 \otimes G_2 = SU(N) \otimes SU(2)$  in which the  $G_1$  sector is of  $G_{ca}$  type rather than the  $G_{cav}$  type studied in the previous section. Two of the simplest cases for the fermion content of the  $SU(N)$  sector involve chiral fermions transforming according to symmetric and antisymmetric rank-2 tensor representations of  $SU(N)$ , denoted  $S_2$  and  $A_2$ , together with the requisite number of fermions in the conjugate fundamental representation. Two minimal anomaly-free  $SU(N)$  sectors are the following, which we shall label as  $S_2 \bar{F}$  and  $A_2 \bar{F}$ :

$$S_2 \bar{F} : \quad \square\square + (N + 4) \bar{\square} \quad \text{for } N \geq 3 \quad (6.1)$$

and

$$A_2\bar{F} : \quad \boxed{\square} + (N-4)\overline{\boxed{\square}} \quad \text{for } N \geq 5. \quad (6.2)$$

We restrict the  $S_2\bar{F}$  theory to have  $N \geq 3$ , since for  $N = 2$  it is a vectorial, rather than chiral, gauge theory. Similarly, we restrict the  $A_2\bar{F}$  theory to have  $N \geq 5$  because for  $N = 4$ , the  $\boxed{\square}$  representation is self-conjugate, so the  $SU(4)$   $A\bar{F}$  theory is a vectorial, rather than chiral, gauge theory. Given the contributions to the  $SU(N)$  triangle anomaly from the fermions in the  $S_2$  and  $A_2$  representations (see Appendix A), these respective  $SU(N)$  theories are anomaly-free. However, we shall show that neither of these can be used to construct an  $N_G = 2$  direct-product chiral gauge theory in which the  $SU(2)$  gauge interaction is asymptotically free.

We form the embeddings of the  $S_2\bar{F}$  and  $A_2\bar{F}$  sectors in an  $SU(N) \otimes SU(2)$  chiral gauge theory with the respective fermion contents

$$(\boxed{\square}, 2) + (N+4)(\overline{\boxed{\square}}, 2) \quad \text{for } N \geq 3 \quad (6.3)$$

and

$$(\boxed{\square}, 2) + (N-4)(\overline{\boxed{\square}}, 2) \quad \text{for } N \geq 5. \quad (6.4)$$

We will denote these as the  $S_2\bar{F}$  and  $A_2\bar{F}$   $SU(N) \otimes SU(2)$  theories respectively, and as the  $T_2\bar{F}$   $SU(N) \otimes SU(2)$  theories (where  $T_2$  stands for rank-2 tensor) when we refer to them together, with  $T_2 = S_2$  or  $A_2$ . These two respective direct-product chiral gauge theories are clearly free of any anomalies in gauged currents. With the respective restrictions on  $N$ , these theories are of type  $(ca, r)$ .

The numbers of  $SU(2)$ -doublet fermions in these two respective  $T_2\bar{F}$   $SU(N) \otimes SU(2)$  theories are

$$N_d = \frac{3N(N \pm 3)}{2}, \quad (6.5)$$

where the upper and lower signs refer to the  $S_2\bar{F}$  and  $A_2\bar{F}$   $SU(N) \otimes SU(2)$  theories respectively. In each case,  $N_d$  must be even in order for the theory to avoid a global  $SU(2)$  anomaly.

The one-loop coefficients in the  $SU(N)$  beta function in these respective theories are

$$b_{1, SU(N)} = \frac{1}{3}(7N \mp 12), \quad (6.6)$$

where again the upper and lower signs refer to the  $S_2\bar{F}$  and  $A_2\bar{F}$   $SU(N) \otimes SU(2)$  theories respectively. In both cases this is positive, so the  $SU(N)$  sector is asymptotically free.

However, the one-loop coefficients in the  $SU(2)$  beta function in the respective theories are

$$b_{1, SU(2)} = \frac{1}{3} \left[ 22 - \frac{3N(N \pm 3)}{2} \right] \quad \text{for } T_2\bar{F}. \quad (6.7)$$

We find that for the  $S_2\bar{F}$   $SU(N) \otimes SU(2)$  theory,  $b_{1, SU(2)}$  is negative for all relevant  $N \geq 3$ . (with  $N$  extended

to the positive real numbers,  $b_{1, SU(2)} < 0$  for  $N > [-9 + \sqrt{609}]/6 = 2.613$ ), so none of these theories has the required asymptotically free  $SU(2)$  gauge interaction. Also, many cases are independently excluded by the fact that  $N_d$  is odd. Regarding the  $A_2\bar{F}$   $SU(N) \otimes SU(2)$  theory, the  $N = 5$  case has a positive  $b_{1, SU(2)}$  (equal to 7), but is excluded because it has an odd value of  $N_d$ , namely  $N_d = 15$ . All other values of  $N$  for the  $A_2\bar{F}$   $SU(N) \otimes SU(2)$  theories are excluded because  $b_{1, SU(2)}$  is negative. (With  $N$  extended to the positive real numbers,  $b_{1, SU(2)} < 0$  for  $N > [9 + \sqrt{609}]/6 = 5.613$ .) Many of these cases are also excluded independently because they have odd  $N_d$ . Therefore, our examination of these  $T_2\bar{F}$   $SU(N) \otimes SU(2)$  theories shows that none of them yields an acceptable chiral gauge theory for our analysis.

## VII. $SU(N_c) \otimes SU(2)_L \otimes U(1)_Y$ THEORIES

Here we shall study the nonperturbative behavior of a chiral gauge theory with a gauge group of the form (1.1) with  $N_G = 3$ , namely

$$G_{GSM} = SU(N_c) \otimes SU(2)_L \otimes U(1)_Y, \quad (7.1)$$

where the subscript GSM stands for “generalized Standard Model”. In this section we will follow an traditional convention in writing some of the fermion fields as right-handed and, related to this, in denoting the  $SU(2)$  gauge group as  $SU(2)_L$ . The fermion content is (with  $i = 1, \dots, N_g$ , where  $N_g$  = number of generations)

$$Q_{i,L}^{a\alpha} = \begin{pmatrix} u_i^a \\ d_i^a \end{pmatrix}_L : \quad N_g(N_c, 2)_{Y_{Q_L}} \quad (7.2)$$

$$(\text{i.e., } Q_{i,L}^{a1} = u_{i,L}^a \text{ and } Q_{i,L}^{a2} = d_{i,L}^a),$$

$$q_{i,R}^a, \quad N_g(N_c, 1)_{Y_{q_R}}, \quad q = u, d \quad (7.3)$$

$$L_{i,L}^\alpha = \begin{pmatrix} \nu_{\ell_i} \\ \ell_i \end{pmatrix}_L : \quad N_g(1, 2)_{Y_{L_L}} \quad (7.4)$$

$$(\text{i.e., } L_{i,L}^1 = \nu_{\ell_i,L} \text{ and } L_{i,L}^2 = \ell_{i,L}),$$

$$\nu_{\ell_i,R}, \quad N_g(1, 1)_{Y_{\nu_R}}, \quad (7.5)$$

and

$$\ell_{i,R} : \quad N_g(1, 1)_{Y_{\ell_R}}. \quad (7.6)$$

Here,  $a$  and  $\alpha$  are color and  $SU(2)_L$  gauge indices, respectively, and  $i$  is a generational index. As listed in Table I, this theory is of type  $(cav, r, c)$ . For our discussion, we will allow the number of colors,  $N_c$ , and  $N_g$  and to be arbitrary, subject to the constraints of asymptotic freedom of the  $SU(N_c)$  and  $SU(2)_L$  gauge interactions and the absence of an  $SU(2)_L$  global anomaly. The capital  $L$  in Eq. (7.4) stands for “lepton” and the subscript  $L$  for the

left-handed chiral component. As in the SM, the (chiral)  $SU(2)_L \otimes U(1)_Y$  gauge group contains a vectorial electromagnetic  $U(1)_{em}$  subgroup, and the electric charge satisfies  $Q_{em} = T_{3L} + (Y/2)$ . Since  $Q_{em, f_L} = Q_{em, f_R} \equiv Q_{em, f}$  for all fermions  $f$ , it follows that the hypercharges of the left-handed and right-handed fermions are related according to

$$Y_{f_R} = 2T_{3L, F_L} + Y_{F_L} , \quad (7.7)$$

where here  $F$  stands for the left-handed quark or lepton  $SU(2)_L$  doublets,  $Q$ ,  $L$ .

This theory is a modification of the Standard Model with the following changes: (i) the color gauge group is changed from  $SU(3)_c$  to  $SU(N_c)$  and (ii)  $N_g$  is arbitrary, both being subject to the three above-mentioned constraints; (iii) the hypercharge assignments are generalized from their real-world values, subject to the constraint that there must not be any gauge anomaly; (iv) two types of  $SU(N_c) \otimes SU(2)_L$ -singlet fermions are present, namely  $\ell_{i,R}$  and  $\nu_{\ell_{i,R}}$ , are present; and (v) the Higgs scalar boson is removed. The  $SU(N_c)$  subsector of this theory is vectorial and hence is free of any anomalies in gauged currents. As before, the  $SU(2)_L$  sector has no pure cubic  $SU(2)_L$  triangle anomaly in gauged currents. Given the structure of this GSM theory, the conditions that there be no triangle anomalies in gauged currents of the form  $SU(N_c)^2 U(1)_Y$  and  $U(1)_Y^3$  are the same. If one imposes the condition that these constraints should be satisfied for each fermion generation individually, as we will (and as is the case in the SM), then the resultant condition is

$$N_c Y_{Q_L} + Y_{L_L} = 0 \quad (7.8)$$

for each fermion generation. The properties of this theory were studied for the usual case  $Y_{L_L} = -1$  in [30] and for general fermion hypercharge assignments in [31]. Provided that the hypercharge assignments satisfy Eq. (7.8), they also yield a vanishing mixed gravitational-gauge anomaly (for each generation) [31]. The generic classes of hypercharge assignments and resultant properties of the theory were given in [31], together with certain special classes. We comment on these further below.

The condition that the  $SU(2)_L$  gauge sector should be free of a global anomaly associated with the homotopy group  $\pi_4(SU(2)) = \mathbb{Z}_2$  is that the number of  $SU(2)_L$  doublets,

$$N_d = N_g(N_c + 1) \quad (7.9)$$

is even, i.e.,

$$N_g(N_c + 1) \text{ is even,} \quad (7.10)$$

and we require that this condition be satisfied. As was noted in [31], if  $N_g$  is even, then this constraint allows arbitrary  $N_c$ , while if  $N_g$  is odd, then it allows only odd  $N_c$ . Similarly, if  $N_c$  is odd, then this constrain allows any value of  $N_g$ , while if  $N_c$  is even, it requires that  $N_g$  to be even.

We note that if one were to turn off the  $U(1)_Y$  gauge interaction and set  $N_c = N$ , then this model would reduce to the special case of Model B in Sect. V with  $p = p' = N_g$  (together with some gauge-singlet fermions). The correspondences between fermion fields in these models is

$$Q_{i,L}^{\alpha} \leftrightarrow \psi_{i,L}^{\alpha}, \quad 1 \leq i \leq p, \quad (7.11)$$

$$\{u_{i,L}^a, d_{i,L}^a\} \leftrightarrow \chi_{a,j,L}, \quad 1 \leq i \leq p; 1 \leq j \leq 2p \quad (7.12)$$

and

$$L_{i,L}^{\alpha} \leftrightarrow \eta_{i,L}^{\alpha}, \quad 1 \leq i \leq p'. \quad (7.13)$$

One reason that we used abstract notation for the fermions in the Models A, B, and C of Sect. V is that they have a different structure than the GSM theory considered here in several respects: (i) the condition for the absence of anomalies in gauge currents is different, since they have no  $U(1)$  factor; and (ii)  $p$  and  $p'$  need not be equal, whereas in the GSM  $p = p' = N_g$ . Since the  $\nu_{i,R}$  and  $\ell_{i,R}$  fields are singlets under  $SU(N_c) \otimes SU(2)_L$ , they have no (nonsinglet) corresponding fields in Model B of Sect. V.

We shall require that both the  $SU(N_c)$  and  $SU(2)_L$  gauge interactions in the GSM must be asymptotically free. The one-loop coefficient of the  $SU(N_c)$  beta function is

$$b_{1,SU(N_c),GSM} = \frac{1}{3}(11N_c - 4N_g), \quad (7.14)$$

so the requirement that the  $SU(N_c)$  gauge interaction must be asymptotically free implies that  $N_g$  satisfies

$$N_g < \frac{11N_c}{4}. \quad (7.15)$$

The one-loop coefficient of the  $SU(2)_L$  beta function is  $b_{1,SU(2),GSM} = (1/3)(22 - N_d)$ , i.e.,

$$b_{1,SU(2),GSM} = \frac{1}{3}[22 - N_g(N_c + 1)], \quad (7.16)$$

so the requirement that the  $SU(2)_L$  gauge interaction must be asymptotically free implies that the number of  $SU(3)_L$  doublets (7.9) is bounded above according to

$$N_g(N_c + 1) < 22. \quad (7.17)$$

The weak hypercharge  $U(1)_Y$  gauge interaction is non-asymptotically free, and the associated gauge coupling  $g'$  decreases as the Euclidean reference scale  $\mu$  decreases. If, as we assume,  $g'$  is weak at a high scale in the UV, then it remains weak at lower scales. Thus, the possible nonperturbative behavior in the theory is due to the growth of the gauge couplings of the non-Abelian gauge interactions.

In our generalized theory, if the  $SU(N_c)$  gauge interaction is sufficiently strong and dominates over the  $SU(2)_L$  interaction, then the former breaks  $G_{EW}$  to  $U(1)_{em}$ , as

in dynamical theories of electroweak symmetry breaking. The most attractive channel is

$$(\bar{N}_c, 2) \times (N_c, 1) \rightarrow (1, 2) \quad (7.18)$$

with attractiveness measure given by (5.15) with  $N = N_c$ . The associated condensates are

$$\langle \bar{Q}_{a,\alpha,i,L} u_{j,R}^a \rangle \quad (7.19)$$

and

$$\langle \bar{Q}_{a,\beta,i,L} d_{j,R}^a \rangle \quad (7.20)$$

(and hermitian conjugates). With no loss of generality, one may choose  $\alpha = 1$  in Eq. (7.19), so that this condensate takes the form  $\langle \bar{u}_{a,i,L} u_{j,R}^a \rangle$ . Since the fermions are massless, one can order the flavor basis of the  $u_{j,R}^a$  fields so that the condensate is diagonal in flavor and hence has the form

$$\langle \bar{u}_{a,i,L} u_{i,R}^a \rangle, \quad i = 1, \dots, N_g. \quad (7.21)$$

This condensate thus breaks the electroweak gauge symmetry according to  $SU(2)_L \otimes U(1)_Y \rightarrow U(1)_{em}$ . As noted in [14], a vacuum alignment argument implies that the condensate (7.20) aligns in a manner so as to preserve this residual  $U(1)_{em}$  gauge symmetry, so that  $\beta = 2$  in (7.20). With an appropriate ordering of the flavor basis of the  $d_{j,R}^a$ , this condensate thus takes the form

$$\langle \bar{d}_{a,i,L} d_{i,R}^a \rangle, \quad i = 1, \dots, N_g. \quad (7.22)$$

If, on the other hand, the  $SU(2)_L$  gauge interaction is sufficiently strong and dominates over the  $SU(N_c)$  gauge interaction, then the gauge symmetry breaking is different. The most attractive channel for the  $SU(2)_L$  interaction is, as before,  $2 \times 2 \rightarrow 1$ . There are three types of condensates that can form in this channel, which we denote for short as  $\langle QQ \rangle$ ,  $\langle QL \rangle$ , and  $\langle LL \rangle$ . These were noted in [2] for the Standard Model without a Higgs field, corresponding to the special case of the GSM with  $N_c = 3$  and  $Y_{LL} = 0$ . Here we extend this analysis to the full GSM. The simplest condensate is  $\langle LL \rangle$ , which has the form

$$\langle \epsilon_{\alpha\beta} L_{i,L}^\alpha C L_{j,L}^\beta \rangle. \quad (7.23)$$

Using the general property (2.5) and taking into account the contraction with  $\epsilon_{\alpha\beta}$ , it follows that the bilinear fermion operator in (7.23) is antisymmetric in the generation indices  $i$  and  $j$ . Hence, if  $N_g = 1$ , it is absent. Assuming  $N_g \geq 2$ , so that the condensate (7.23) forms, it preserves the  $SU(N_c) \otimes SU(2)_L$  part of  $G_{GSM}$  and, for all but a set of measure zero of hypercharge assignments, it breaks the  $U(1)_Y$  weak hypercharge gauge symmetry, transforming as a  $\Delta Y = 2Y_{LL}$  operator. The only exception is the case denoted class  $C2_{\ell,sym} = C2_{q,sym}$  in [31] (see Tables I and II in [31]), for which  $Y_{LL} = 0 = Y_{QL}$ . The condensate (7.23) also breaks the (global) lepton family number  $U(1)_{L_i}$  and total lepton number  $U(1)_L$

symmetries. However, these global symmetries are already broken by  $SU(2)_L$  instantons, and since we assume that  $SU(2)_L$  is strongly coupled, these  $SU(2)_L$  instantons are not suppressed as they are (at zero temperature) in the Standard Model. Note that if one assumes conventional weak hypercharge assignments, so that  $\nu_{i,R}$  fermions are GSM-singlets, then  $\nu_{i,R}^T C \nu_{j,R}$  Majorana mass terms are, in general, present, and explicitly break both lepton family number and total lepton number.

The second type of condensate, denoted  $\langle QL \rangle$ , has the form

$$\langle \epsilon_{\alpha\beta} Q_{i,L}^{\alpha\alpha} C L_{j,L}^\beta \rangle. \quad (7.24)$$

where  $1 \leq i, j \leq N_g$ . This is analogous to the condensate (5.32) in the  $SU(N) \otimes SU(2)_L$  Model B of Section V, with the correspondence  $N = N_c$  and  $p = p' = N_g$ , so our analysis in that section applies here, with these identifications. In particular, if  $N_g \leq N_c - 2$ , then this set of condensates breaks  $SU(N_c)$  down to  $SU(N_c - N_g)$ , while if  $N_g \geq N_c - 1$ , then this set of condensates breaks  $SU(N_c - N_g)$  completely. These condensates also break baryon number and (total and family) lepton number.

The third type of condensate, denoted  $\langle QQ \rangle$ , has the form

$$\langle \epsilon_{\alpha\beta} Q_{i,L}^{\alpha\alpha} C Q_{j,L}^{\beta\beta} \rangle. \quad (7.25)$$

The same analysis that we gave above for the condensate (5.17) in the  $SU(N) \otimes SU(2)$  gauge theory applies here, with  $N = N_c$  and  $p = N_g$ . From this analysis we infer that the condensation channel is Eq. (5.21) with  $N = N_c$ , and the type of  $\langle QQ \rangle$  condensate that is produced here is

$$\begin{aligned} & \langle \epsilon_{\alpha\beta} (Q_{i,L}^{\alpha,a} C Q_{j,L}^{\beta,b} - Q_{i,L}^{\alpha,b} C Q_{j,L}^{\beta,a} \\ & + Q_{j,L}^{\alpha,a} C Q_{i,L}^{\beta,b} - Q_{j,L}^{\alpha,b} C Q_{i,L}^{\beta,a}) \rangle. \end{aligned} \quad (7.26)$$

This is invariant under  $SU(2)_L$  and breaks  $SU(N_c)$  according to Eq. (5.22) with  $N = N_c$ . For all but a set of measure zero of weak hypercharge assignments, the condensate (7.26) also breaks  $U(1)_Y$ , transforming as a  $\Delta Y = 2Y_{QL}$  operator. The sole exception is the case where  $Y_{QL} = 0 = Y_{LL}$ , denoted as class  $C2_{q,sym} = C2_{\ell,sym}$  in [31] (see Tables I and II in [31]). The condensate (7.26) also breaks baryon number,  $U(1)_B$ , but, as noted, this is already broken by the  $SU(2)_L$  instantons.

### VIII. $SU(N_c) \otimes U(1)_Y$ THEORIES WITH $N_c \geq 3$

In this and the next two sections we shall apply the reduction procedure discussed in Section III A to obtain two (anomaly-free)  $N_G = 2$  chiral gauge theories starting with the generalized Standard Model theory discussed in Section VII. These are obtained by turning off the  $SU(2)_L$  coupling and the  $SU(N_c)$  coupling, respectively.

The third possibility, namely to turn off the  $U(1)_Y$  coupling, yields a theory with the group  $SU(N_c) \otimes SU(2)_L$ , which was already analyzed in Section V.

We begin by turning off the  $SU(2)_L$  coupling in the generalized Standard Model, thereby obtaining the gauge group

$$G = SU(N_c) \otimes U(1)_Y \quad (8.1)$$

with the (nonsinglet) fermion content given by Eqs. (7.2) and (7.3). This theory is of the type  $(cav, ca)$  in the classification of Section II. As in the GSM itself, because the  $SU(N_c)$  gauge interaction is vectorial, the  $SU(N_c)^3$  anomaly is zero. In the GSM,  $Y_{Q_L}$  denotes the generalized weak hypercharge of the left-handed quark doublet in Eq. (7.2); here, since the theory does not have any  $SU(2)_L$ , we take it simply to be the common value of  $Y$  for  $u_{i,L}^a$  and  $d_{i,L}^a$  (and the same for all  $i = 1, \dots, N_g$ ). Because the original GSM contains a vectorial  $U(1)_{em} \subset SU(2)_L \otimes U(1)_Y$ , which yields the relation (7.7), it follows in the present truncated model that if we specify  $Y_{Q_L}$ , then the hypercharges  $Y_{u_R}$  and  $Y_{d_R}$  are determined. Thus, just as was true in the GSM, as discussed in [31], in this truncated version, there is actually an infinite one-parameter family of models that depend, here, on  $Y_Q$ . For any member of this family, as a special case of the situation in the GSM, it follows that the theory is free of (i) any  $SU(N_c)^2 U(1)_Y$  triangle anomaly, (ii) any  $U(1)_Y^3$  anomaly, and (iii) any mixed gravitational-gauge anomaly involving the  $U(1)_Y$  gauge group.

The one-loop coefficient for the  $SU(N_c)$  beta function is given by Eq. (7.14), so the upper bound on  $N_g$  to ensure the asymptotic freedom of the  $SU(N_c)$  gauge interaction is the same as in (7.15). As the theory evolves from the UV to the IR and the  $SU(N_c)$  gauge couplings gets sufficiently large, the theory forms bilinear quark condensates in the  $SU(N_c)$  MAC, which is  $\square \times \square \rightarrow 1$ . *A priori*, these condensates would be

$$\langle \bar{u}_{a,i,L} u_{j,R}^a \rangle, \langle \bar{d}_{a,i,L} d_{j,R}^a \rangle, \langle \bar{u}_{a,i,L} d_{j,R}^a \rangle, \langle \bar{d}_{a,i,L} u_{j,R}^a \rangle \quad (8.2)$$

(and their hermitian conjugates). However, a vacuum alignment argument can be used to infer that the condensate formation is such as to preserve the  $U(1)_{em}$  subgroup of the  $U(1)_Y$  gauge symmetry, i.e., only the  $\langle \bar{u}_{a,i,L} u_{i,R}^a \rangle$  and  $\langle \bar{d}_{a,i,L} d_{i,R}^a \rangle$  condensates (and their hermitian conjugates) form. Since the theory has no bare mass terms, for a fixed ordering of the generational indices of the left-handed quarks  $u_{i,L}^a$  and  $d_{i,L}^a$ , we can always choose the order of the the generational indices of the  $u_{j,R}^a$  and  $d_{j,R}^a$  so that the condensates are diagonal in generation indices. The  $\langle \bar{u}_{a,i,L} u_{i,R}^a \rangle$  and  $\langle \bar{d}_{a,i,L} d_{i,R}^a \rangle$  condensates each break  $U(1)_Y$  to  $U(1)_{em}$ .

## IX. $SU(2) \otimes U(1)_Y$ THEORIES

Here we obtain a chiral gauge theory of the type  $(r, ca)$  by starting with the generalized Standard Model of

Section VII and turning off the  $SU(N_c)$  gauge coupling, thereby obtaining the gauge group

$$SU(2) \otimes U(1)_Y. \quad (9.1)$$

The fermions are given by (7.2) and (7.4) of the GSM, with the modification that now the color index is a global, rather than gauge, index. The condition that the  $SU(2)$  theory must not have any global anomaly is the same as Eq. (7.10), and, as in the GSM, if one imposes it individually on each generation, then it is the statement that  $N_c$  must be odd.

The one-loop coefficient in the  $SU(2)_L$  beta function is the same as in Eq. (7.16), and the resultant upper bound on  $N_g(N_c + 1)$  resulting from the condition that the  $SU(2)_L$  gauge interaction must be asymptotically free is thus the same as in (7.17). As the theory evolves from the UV to the IR and the  $SU(2)$  grows, if it becomes sufficiently large, it can produce condensates in the  $SU(2)$  MAC,  $2 \times 2 \rightarrow 1$ , of the three forms discussed in Section VII, denoted for short as  $\langle LL \rangle$ ,  $\langle QL \rangle$ , and  $\langle QQ \rangle$ , with associated condensates (7.23), (7.24), and (7.26). As discussed in Section VII, except for a set of measure zero, namely the case where  $Y_{Q_L} = Y_{L_L} = 0$ , denoted  $C2_{q,sym} = C2_{\ell,sym}$  in [31], these condensates break  $U(1)_Y$ .

## X. $SU(N) \otimes SU(2)_L \otimes SU(2)_R$ THEORIES WITH $N \geq 3$

In this section we consider another chiral gauge theory with a gauge group of the form (1.1) with  $N_G = 3$ , namely

$$G_{N22} = SU(N) \otimes SU(2)_L \otimes SU(2)_R \quad (10.1)$$

with  $N \geq 3$  and the fermions

$$\psi_{i,L}^{a,\alpha_L}, i = 1, \dots, p : p(\square, \square, 1) = p(N, 2, 1), \quad (10.2)$$

$$\psi_{i,R}^{a,\alpha_R}, i = 1, \dots, p : p(\square, 1, \square) = p(N, 1, 2), \quad (10.3)$$

$$\chi_{j,L}^{\alpha_L}, j = 1, \dots, p' : p'(1, \square, 1) = p'(1, 2, 1), \quad (10.4)$$

and

$$\chi_{j,R}^{\alpha_R}, j = 1, \dots, p'' : p''(1, 1, \square) = p''(1, 1, 2). \quad (10.5)$$

Here the three representations in the parentheses refer, respectively, to the three factor groups in Eq. (10.1). As indicated in Table I, this theory is of type  $(cav, r, r)$ . Since the  $SU(N)$  gauge interaction is vectorial, it has no gauge anomaly, and both the  $SU(2)_L$  and  $SU(2)_R$  gauge sectors are safe (anomaly-free). The conditions that the  $SU(2)_L$  and  $SU(2)_R$  gauge sectors should be free of a global anomaly are, respectively,

$$pN + p' \text{ is even} \quad (10.6)$$

and

$$pN + p'' \text{ is even.} \quad (10.7)$$

As with our other models, we shall require that all three non-Abelian gauge interactions are asymptotically free. The one-loop coefficient of the  $SU(N)$  beta function is the same as in the  $SU(N) \otimes SU(2)_L$  model of Sect. V, Eq. (5.9) (applicable to both Models A and B of that section) so the condition that the  $SU(N)$  gauge interaction should be asymptotically free is the inequality (5.10). The one-loop coefficient of the  $SU(2)_L$  beta function is the same as in the  $SU(N) \otimes SU(2)_L$  Model B, Eq. (5.27), so the requirement that the  $SU(2)_L$  gauge interaction be asymptotically free is the inequality (5.28). Finally, the one-loop coefficient of the  $SU(2)_R$  beta function is the same as Eq. (5.27) with  $p'$  replaced by  $p''$ , so the requirement that the  $SU(2)_R$  gauge interaction be asymptotically free is given by the inequality (5.28) with  $p'$  replaced by  $p''$ , namely  $Np + p'' < 22$ .

We denote the gauge couplings as  $g_N$ ,  $g_L$ , and  $g_R$ , with  $\alpha_N = g_N^2/(4\pi)$ ,  $\alpha_L = g_L^2/(4\pi)$ , and  $\alpha_R = g_R^2/(4\pi)$ . If the initial values of these couplings are such that, as the Euclidean reference scale  $\mu$  decreases from large values in the deep UV, the  $SU(N)$  interaction becomes sufficiently strong and dominates over the  $SU(2)_L$  and  $SU(2)_R$  gauge interactions, then it is expected to produce condensation in the most attractive channel, which is

$$(\bar{N}, 2, 1) \times (N, 1, 2) \rightarrow (1, 2, 2), \quad (10.8)$$

with attractiveness measure (5.15). The associated bilin-

ear fermion condensate is

$$\langle \bar{\psi}_{a,\alpha_L,i,L} \psi_{j,R}^{a,\alpha_R} \rangle. \quad (10.9)$$

This breaks  $SU(2)_L \otimes SU(2)_R$  gauge symmetry to the diagonal (= vector) subgroup,  $SU(2)_V$ . That is, if elements of  $SU(2)_L$  and  $SU(2)_R$  are denoted as  $U_L$  and  $U_R$ , then  $SU(2)_V$  is the subgroup of  $SU(2)_L \otimes SU(2)_R$  defined by the condition  $U_L = U_R$ .

If the  $SU(2)_L$  interaction is sufficiently strong and dominates over both the  $SU(N)$  and  $SU(2)_R$  interaction, then it can produce the three types of condensates and corresponding symmetry breaking discussed in our analysis of the  $SU(N) \otimes SU(2)_L$  Model B above.

Finally, if the  $SU(2)_R$  interaction is sufficiently strong and dominates over both the  $SU(N)$  and  $SU(2)_L$  interaction, our discussion of the condensate formation in the  $SU(N) \otimes SU(2)_L$  Model B above applies, with all subscripts  $L$  changed to  $R$ .

## XI. $SU(N_c) \otimes SU(2)_L \otimes SU(2)_R \otimes U(1)_{B-L}$ THEORIES

Here we analyze the chiral gauge theory with a gauge group of the form (1.1) with  $N_G = 4$ , namely

$$G_{N221} = SU(N_c) \otimes SU(2)_L \otimes SU(2)_R \otimes U(1)_{B-L}. \quad (11.1)$$

We denote the gauge couplings as  $g_{N_c}$ ,  $g_L$ ,  $g_R$ , and  $g_U$ , with  $\alpha_{N_c} = g_{N_c}^2/(4\pi)$ , and so forth for the other couplings. The quarks and leptons in this theory are

$$Q_{i,L}^{a,\alpha_L}, \quad i = 1, \dots, N_g : N_g(\square, \square, 1)_{1/N_c} = N_g(N_c, 2, 1)_{1/N_c}, \quad (11.2)$$

$$Q_{i,R}^{a,\alpha_R}, \quad i = 1, \dots, N_g : N_g(\square, 1, \square)_{1/N_c} = N_g(N_c, 1, 2)_{1/N_c}, \quad (11.3)$$

$$L_{i,L}^{\alpha_L}, \quad i = 1, \dots, N_g : N_g(1, \square, 1)_{-1} = N_g(1, 2, 1)_{-1}, \quad (11.4)$$

and

$$L_{i,R}^{\alpha_R}, \quad i = 1, \dots, N_g : N_g(1, 1, \square)_{-1} = N_g(1, 1, 2)_{-1}. \quad (11.5)$$

Here the three numbers in the parentheses are the dimensionalities of the  $SU(N_c)$ ,  $SU(2)_L$ , and  $SU(2)_R$  representations, and the subscripts are the value of  $B - L$ , where  $B$  and  $L$  denote baryon and lepton number. The capital  $L$  in Eqs. (11.4) and (11.5) stands for “lepton” and the subscripts  $L$  and  $R$  for left- and right-handed chiral components, as before. This theory is of type  $(cav, r, r, cav)$  (see Table I) and is a modification of the model of Ref. [33] in that (i) the number of colors,  $N_c \geq 3$  and (ii) the

number of generations,  $N_g$ , are arbitrary, subject to constraints to be discussed below; and (iii) the Higgs field is removed. One of the interesting features of the original model of Ref. [33] is that the  $B - L$  operator applied to the full set of quarks and leptons in each generation has zero trace. Our generalized model retains this property, since  $B = 1/N_c$  for each quark. A second interesting feature of the original model is that electric charge  $Q_{em} = T_{3L} + T_{3R} + (B - L)/2$  is quantized, since  $T_{3L}$ ,

$T_{3R}$ ,  $B$ , and  $L$  are rational (indeed,  $L$  is integral). Again, our generalized model retains this feature.

The  $SU(N_c)$  gauge interaction is vectorial, and hence has no gauge anomaly, and both the  $SU(2)_L$  and  $SU(2)_R$  gauge sectors are also free of any pure cubic gauge anomalies. The theory is also free of  $SU(2)_L^2 U(1)_{B-L}$ ,  $SU(2)_R^2 U(1)_{B-L}$ , and  $U(1)_{B-L}^3$  triangle gauge anomalies. The theory is also free of any mixed gravitational-gauge anomaly. The conditions that the  $SU(2)_L$  and  $SU(2)_R$  gauge sectors are each free of any global anomaly are the same, namely the condition (7.10).

We shall require that the three non-Abelian gauge interactions be asymptotically free. The one-loop coefficient of the  $SU(N_c)$  beta function is the same as in the generalized Standard Model, Eq. (7.14), so the condition that the  $SU(N_c)$  gauge interaction must be asymptotically free is the same as the inequality (7.15). The respective one-loop coefficients of the  $SU(2)_L$  and  $SU(2)_R$  beta functions are equal to each other and given by Eq. (7.16), so the condition that the  $SU(2)_L$  and  $SU(2)_R$  gauge interactions must be asymptotically free is the same as the inequality (7.17). The  $U(1)_{B-L}$  gauge interaction is non-asymptotically free, and the associated gauge coupling  $g_U$  decreases with decreasing scale  $\mu$ . If, as we assume,  $g_U$  is weak at a high scale in the UV, then it remains weak at lower scales. Thus, the possible nonperturbative behavior in the theory is due to the growth of the gauge couplings of the three non-Abelian gauge interactions.

If the initial values of these couplings are such that, as the Euclidean reference scale  $\mu$  decreases from large values in the deep UV, the  $SU(N_c)$  interaction becomes sufficiently strong and dominates over the  $SU(2)_L$  and  $SU(2)_R$  gauge interactions, then it is expected to produce condensation in the most attractive channel, which is

$$(\bar{N}_c, 2, 1)_{-1/N_c} \times (N_c, 1, 2)_{1/N_c} \rightarrow (1, 2, 2)_0. \quad (11.6)$$

The associated bilinear fermion condensate is the same as the one given in Eq. (10.9). As is evident from (11.6), this preserves the  $SU(N_c)$  and  $U(1)_{B-L}$  gauge symmetries and breaks  $SU(2)_L \otimes SU(2)_R$  to  $SU(2)_V$ .

If the  $SU(2)_L$  interaction is sufficiently strong and dominates over both the  $SU(N_c)$  and  $SU(2)_R$  interaction, then it can produce the three types of condensates discussed in our analysis of the generalized Standard Model above, with appropriate changes of weak hypercharge to  $B-L$ . The first of these is the condensate denoted (7.23) with the replacements  $\alpha, \beta \rightarrow \alpha_L, \beta_L$  and our discussion in connection with this condensate applies here. In particular, assuming  $N_g \geq 2$ , so that this condensate forms,

it preserves the  $SU(N_c) \otimes SU(2)_L \otimes SU(2)_R$  part of  $G_{N221}$  and breaks the  $U(1)_{B-L}$  gauge symmetry, transforming as  $|\Delta L| = 2$ .

The second type of condensate has the form of (7.24) with  $i, j = 1, \dots, N_g$ . This condensate is invariant under the  $SU(2)_L \otimes SU(2)_R$  part of  $G_{N221}$  and, for a given  $i, j$ , it breaks  $SU(N_c)$  to  $SU(N_c - 1)$ . Without loss of generality, we may choose  $a = N_c$ , so that the residual subgroup  $SU(N_c - 1)$  operates on the indices  $a \in \{1, \dots, N_c - 1\}$ . It also breaks  $U(1)_{B-L}$ , since it transforms as an operator with  $|B - L| = |N_c^{-1} - 1| \neq 0$ .

The third type of condensate is  $\langle QQ \rangle$ , which has the form of (7.25) with  $\alpha, \beta \rightarrow \alpha_L, \beta_L$ . The same analysis that we gave above for this condensate in our discussion of the generalized Standard Model applies here, with the obvious change of  $\alpha, \beta$  just noted. Thus, again, using MAC and vacuum alignment arguments, we may infer that the condensate has the explicit structure of Eq. (7.26). This is invariant under the  $SU(2)_L \otimes SU(2)_R$  part of  $G_{N221}$  and breaks  $SU(N_c)$  according to Eq. (5.22) with  $N = N_c$ . It also breaks  $U(1)_{B-L}$ , transforming as a  $|\Delta B| = 2/N_c$  operator.

Finally, if the  $SU(2)_R$  interaction is sufficiently strong and dominates over both the  $SU(N_c)$  and  $SU(2)_L$  interaction, then it can produce the three types of condensates discussed directly above, with the obvious changes of chiralities of fermion fields from  $L$  to  $R$  and the resultant changes of symmetry-breaking patterns.

## XII. $SU(N_c + 1) \otimes SU(2)_L \otimes SU(2)_R$ THEORIES

As noted above, in the original model with an  $SU(3)_c \otimes SU(2)_L \otimes SU(2)_R \otimes U(1)_{B-L}$  electroweak gauge group [33], the  $B - L$  operator applied to the full set of quarks and leptons in each generation has zero trace. Owing to this property, one can embed the  $U(1)_{B-L}$  gauge symmetry together with  $SU(3)_c$  in an  $SU(4)$  group [34] such that the  $B - L$  operator  $\text{diag}(1/3, 1/3, 1/3, -1)$  is proportional to the last diagonal generator of the Cartan subalgebra of  $\mathfrak{su}(4)$ . The resultant gauge group is  $SU(4) \otimes SU(2)_L \otimes SU(2)_R$ . We may carry out the same process for our generalized group and thus consider the chiral gauge theory with gauge group

$$G = SU(N_c + 1) \otimes SU(2)_L \otimes SU(2)_R. \quad (12.1)$$

The fermion content is

$$F_L = (Q_i^{a, \alpha_L}, L_i^{\alpha_L})_L, \quad i = 1, \dots, N_g : \quad N_g(\square, \square, 1) = N_g(N_c + 1, 2, 1), \quad (12.2)$$

$$F_R = (Q_i^{a, \alpha_R}, L_i^{\alpha_R})_R, \quad i = 1, \dots, N_g : \quad N_g(\square, 1, \square) = N_g(N_c + 1, 1, 2), \quad (12.3)$$

where,  $a$ ,  $\alpha_L$ , and  $\alpha_R$  are, respectively,  $SU(N_c + 1)$ ,  $SU(2)_L$ , and  $SU(2)_R$  gauge indices and  $i$  is a generation index. The three numbers in the parentheses are the dimensionalities of the  $SU(N_c + 1)$ ,  $SU(2)_L$ , and  $SU(2)_R$  representations. The Cartan subalgebra of  $\mathfrak{su}(N_c + 1)$  has dimension  $N_c + 1$  and its last Cartan matrix is proportional to a diagonal matrix whose first  $N_c$  entries are  $1/N_c$  and whose  $N_c + 1$ 'th entry is  $-1$ , i.e.,  $\text{diag}(1/N_c, \dots, 1/N_c, -1)$ .

We observe that this model is a special case of the chiral gauge theory that we analyzed in Section X obtained by setting  $N = N_c + 1$ ,  $p = N_g$ ,  $p' = p'' = 0$ ,  $\psi_{i,L}^{a,\alpha_L} = F_L$ , and  $\psi_{i,R}^{a,\alpha_R} = F_R$ . Thus, this special case of our analysis in Section X applies for the theory of this section.

### XIII. $SO(4k + 2) \otimes SU(2)$ THEORIES

It is also of interest to study chiral gauge theories with direct-product groups that involve a safe  $SO(N)$  group. We recall that if  $N$  is odd or if  $N$  is even and  $N = 4k$ ,  $k \geq 1$ , then  $SO(N)$  has only real representations, while if  $N = 4k + 2$  with  $k \geq 2$ , then the theory has complex representations but is safe (i.e., has no anomaly for any representation) [22]. With this motivation, we consider chiral gauge theories with the gauge group

$$G = SO(4k + 2) \otimes SU(2) \quad \text{with } k \geq 2. \quad (13.1)$$

These are of the form  $(cs, cs)$  in the general classification given in Section II. Since  $N$  is even, it is also convenient to introduce an integer  $r = N/2$ :

$$N = 4k + 2 = 2r, \quad k \geq 2, \quad (13.2)$$

so  $r = 2k + 1$ . As before, we write all fermions as left-handed. We start by considering the general fermion content

$$\sum_{\mathcal{R}, \mathcal{R}'} \left[ n_{\mathcal{R}}(\mathcal{R}, 1) + p(\mathcal{R}', \square) \right], \quad (13.3)$$

where  $\mathcal{R}$  and  $\mathcal{R}'$  are representations of  $SO(4k + 2)$ . We include only complex  $\mathcal{R}$  and  $\mathcal{R}'$  since the use of a real  $\mathcal{R}$  or  $\mathcal{R}'$  would lead to a vectorlike subsector, so the model would not be irreducibly chiral.

Using the relevant group invariants, we calculate the one-loop term in the beta function for the  $SO(N)$  gauge coupling with  $N$  given by (13.2) to be

$$b_{SO(4k+2),1} = \frac{2}{3} \left[ 11(r - 1) - \sum_f \left( n_{\mathcal{R}} T_{\mathcal{R}} + 2p_{\mathcal{R}'} T_{\mathcal{R}'} \right) \right]. \quad (13.4)$$

We calculate the one-loop term in the  $SU(2)$  beta function to be

$$b_{SU(2),1} = \frac{1}{3} \left[ 22 - 2 \sum_{\mathcal{R}'} p_{\mathcal{R}'} \dim(\mathcal{R}') \right]. \quad (13.5)$$

Because the first terms in square brackets in Eq. (13.4) and (13.5) are, respectively, linear in  $r$  and a constant, while the relevant  $T_{\mathcal{R}}$ ,  $T_{\mathcal{R}'}$ , and  $\dim(\mathcal{R}')$  grow exponentially rapidly with  $r$ , the asymptotic freedom of the  $SO(2r)$  and  $SU(2)$  gauge interactions places strong restrictions on the fermion content and the value of  $N$ . For our purposes, it will be sufficient to consider the simplest models of this type, with (complex)  $\mathcal{R} = \mathcal{R}'$ . We will consider three specific models, which we label Models A, B, and C.

#### A. Model A

We first briefly consider the case where the fermion sector has the form  $\{f_{ns,s}\}$ , i.e., all of the fermions are singlets under  $SU(2)$ . In this case, the gauge group effectively reduces to  $SO(N)$ , with  $N$  given by (13.2). We choose the minimal complex representation for the fermions, namely the spinor representation, denoted  $\mathcal{S}$ , of dimension  $\dim(\mathcal{S}) = 2^{r-1} = 2^{2k}$  (see Appendix A) and include  $n$  copies of these, so the fermion content is

$$\omega_{i,L}, \quad i = 1, \dots, n : \quad n(\mathcal{S}, 1), \quad (13.6)$$

where the first and second entries in the parentheses here and below are the representations of  $SO(N)$  and  $SU(2)$ , respectively. The general formula for the one-loop term in the beta function for the  $SO(N)$  gauge coupling, Eq. (13.4) for this Model A reduces to

$$b_{SO(2r),1} = \frac{2}{3} \left[ 11(r - 1) - 2^{r-4} n \right]. \quad (13.7)$$

The requirement that the  $SO(N)$  gauge interaction should be asymptotically free implies that

$$n < \frac{11(r - 1)}{2^{r-4}}. \quad (13.8)$$

This has only a finite number of solutions for  $n$  that are nontrivial, i.e., have  $n \geq 1$ , and, indeed, also a finite number of solutions for  $r$ .

$$G_1 = SO(10) \text{ (i.e., } k = 2, r = 5) \Rightarrow n \leq 21 \quad (13.9)$$

$$G_1 = SO(14) \text{ (i.e., } k = 3, r = 7) \Rightarrow n \leq 8 \quad (13.10)$$

$$G_1 = SO(18) \text{ (i.e., } k = 4, r = 9) \Rightarrow n \leq 2 \quad (13.11)$$

For  $k \geq 5$ , i.e.,  $r \geq 11$ , the upper bound on  $n$  is less than unity, precluding any fermions.

We assume some initial value of the  $SO(2r)$  gauge coupling in the deep UV and then evolve the theory downward in Euclidean scale  $\mu$ . Recall that the direct product of two spinor representations of  $SO(N)$  with  $N$  given by (13.2) is [23]

$$\mathcal{S} \times \mathcal{S} = 2^{2k} \times 2^{2k} = \sum_{\ell=0}^{k-1} A_{2\ell+1} + R_{2k+1;2}, \quad (13.12)$$



where  $A_t$  denotes the rank- $t$  antisymmetric tensor representation and  $R_{2k+1}$  is a certain self-dual representation. The symmetry of the  $A_t$  with respect to the interchange of the two spinor representations in the direct product is given by  $(-1)^{u(r,t)}$ , where  $u(r,t) = (r-t)(r-t-1)/2$  [23]. Thus, for example, one has, for the lowest relevant value of  $k$ , namely  $k = 2$ , i.e.,  $G_1 = \text{SO}(10)$ ,

$$\begin{aligned} \text{SO}(10) : \quad \mathcal{S} \times \mathcal{S} &= 2^4 \times 2^4 = A_1 + A_3 + R_{5;2} \\ &= 10_s + 120_a + 126_s, \end{aligned} \quad (13.13)$$

where the subscripts  $s$  and  $a$  denote the symmetric and antisymmetric property of these representations under interchange of the spinors in the direct product. In general, for  $\text{SO}(2k+2)$ , from the form of  $u(r,t)$ , it follows that  $A_1$  is symmetric (resp. antisymmetric) under interchange of the spinors in the direct product for even  $k$  (resp. odd  $k$ ), while  $A_3$  is antisymmetric (resp. symmetric) under interchange of these spinors for even  $k$  (resp. odd  $k$ ).

Assuming that the  $\text{SO}(N)$  coupling becomes strong enough to produce a bilinear fermion condensate, the MAC is

$$\text{SO}(N) \text{ MAC} : \quad \mathcal{S} \times \mathcal{S} \rightarrow \square, \quad (13.14)$$

with attractiveness measure (written, for convenient reference, in terms of each of the three related parameters  $N$ ,  $r$ , and  $k$ )

$$\begin{aligned} \Delta C_2 &= 2C_2(\mathcal{S}) - C_2(\square) = \frac{(N-1)(N-4)}{8} \\ &= \frac{(2r-1)(r-2)}{4} = \frac{(4k+1)(2k-1)}{4}. \end{aligned} \quad (13.15)$$

Since  $r \geq 5$ , i.e.,  $k \geq 2$ , this is always positive. The associated condensate is  $\langle \omega_{i,L}^T C \omega_{j,L} \rangle$ , where  $1 \leq i, j \leq n$ . From the general result (2.5), it follows that the bilinear fermion operator  $\omega_{i,L}^T C \omega_{j,L}$  in this condensate is (i) symmetric under interchange of spinors in the  $\mathcal{S} \times \mathcal{S}$  direct product in (13.14) and hence symmetric in the flavor indices  $i, j$  if  $k$  is even; (ii) antisymmetric under interchange of spinors and hence antisymmetric in the flavor indices  $i, j$  if  $k$  is odd. Therefore, explicitly,

$$k \text{ even} \Rightarrow \langle \omega_{i,L}^T C \omega_{j,L} + \omega_{j,L}^T C \omega_{i,L} \rangle, \quad 1 \leq i, j \leq n \quad (13.16)$$

and

$$k \text{ odd} \Rightarrow \langle \omega_{i,L}^T C \omega_{j,L} - \omega_{j,L}^T C \omega_{i,L} \rangle, \quad 1 \leq i, j \leq n. \quad (13.17)$$

In both cases, if this condensate forms, then, since it transforms as the fundamental (vector) representation of the gauge group  $\text{SO}(4k+2)$ , it breaks this symmetry to  $\text{SO}(4k+1)$ , which is vectorial and does not break further.

However, if  $n = 1$  and  $k$  is odd, e.g., for  $\text{SO}(14)$  (i.e.,  $k = 3$ ), then this condensate in the MAC channel vanishes identically. In this case, we consider the next channel in Eq. (13.12), namely

$$\mathcal{S} \times \mathcal{S} \rightarrow A_3 \quad (13.18)$$

with attractiveness measure

$$\begin{aligned} \Delta C_2 &= 2C_2(\mathcal{S}) - C_2(A_3) = \frac{(N-4)(N-9)}{8} \\ &= \frac{(r-2)(2r-9)}{4} = \frac{(2k-1)(4k-7)}{4}. \end{aligned} \quad (13.19)$$

For the relevant value of  $k$ , namely  $k = 3$ , this is  $\Delta C_2 = 25/4$ .

## B. Model B

Here we consider a model with the gauge group (13.1) with (13.2) and fermion content of the form  $\{f_{ns,ns}\}$ , namely

$$\psi_{i,L}^\alpha, \quad i = 1, \dots, p : \quad p(\mathcal{S}, \square). \quad (13.20)$$

We denote this as Model B. Since there are an even number of  $\text{SU}(2)$  doublets, this theory has no global  $\text{SU}(2)_L$  anomaly.

The general formulas for the one-loop coefficients in the  $\text{SO}(N)$  beta function (with  $N$  given by (13.2)) and in the  $\text{SU}(2)$  beta function displayed in Eqs. (13.4) and (13.5) reduce, for this Model B, to

$$b_{\text{SO}(2r),1} = \frac{2}{3}[11(r-1) - 2^{r-3}p] \quad (13.21)$$

and

$$b_{1,\text{SU}(2)_L} = \frac{2}{3}(11 - 2^{r-2}p). \quad (13.22)$$

Hence, the respective conditions that the  $\text{SO}(2r)$  and  $\text{SU}(2)$  gauge interactions should be asymptotically free are

$$p < \frac{11(r-1)}{2^{r-3}} \quad (13.23)$$

and

$$p < \frac{11}{2^{r-2}}. \quad (13.24)$$

Since we take  $k \geq 2$ , i.e.,  $r \geq 5$ , for our theories, the only possible nontrivial value for  $p$  allowed by the constraint (13.24) is  $p = 1$  and, furthermore, this is only possible for the lowest value of  $k$ , namely  $k = 2$ , and thus  $G_1 = \text{SO}(10)$ . No  $\text{SO}(4k+2)$  theories of this Model B type with nonzero fermion content are allowed by the asymptotic freedom constraint if  $k \geq 3$ .

We note that there is consequently no (continuous) nonanomalous global flavor symmetry of the Lagrangian for this theory. Since there is only one copy of the  $(\mathcal{S}, \square)$  fermion  $\psi_{i,L}^\alpha$ , we shall henceforth drop the flavor index and write this field simply as  $\psi_L^\alpha$ .

If the  $\text{SO}(10)$  gauge interaction is sufficiently strong and dominates over the  $\text{SU}(2)$  gauge interaction, then it produces a condensate in the  $\text{SO}(10)$  MAC, (13.14), thereby breaking the  $\text{SO}(10)$  gauge symmetry to  $\text{SO}(9)$ , which is vectorial and does not break further. The condensate is  $\langle \psi_L^\alpha {}^T C \psi_L^\beta \rangle$ . As noted above in Section XIII A, for  $\text{SO}(4k+2)$ , the  $\square = A_1$  that occurs in the Clebsch-Gordan decomposition of the direct product  $\mathcal{S} \times \mathcal{S}$  in (13.14) is symmetric (resp. antisymmetric) under interchange of these spinors if  $k$  is even (resp. odd). Since  $k=2$  is even here, it follows that this  $\square$  representation is symmetric under interchange of the spinors in the direct product. From the property (2.5), it then follows that the  $\text{SU}(2)$  gauge indices must also be symmetric, i.e., the  $\text{SU}(2)$  channel is  $2 \times 2 \rightarrow 3_s$ , so the operator product transforms as the adjoint (equivalently, the rank-2 symmetric tensor) representation of  $\text{SU}(2)$  and hence can be written as proportional to

$$\langle \psi_L^\alpha {}^T C \psi_L^\beta + \psi_L^\beta {}^T C \psi_L^\alpha \rangle. \quad (13.25)$$

Hence, including both factor groups, in this case of a strong and dominant  $\text{SO}(10)$  gauge interaction with even  $k$  (viz.,  $k=2$ ), the condensation is in the channel

$$k \text{ even} \Rightarrow (\mathcal{S}, \square) \times (\mathcal{S}, \square) \rightarrow (\square_s, \text{adj}_s) = ((4k+2)_s, 3_s). \quad (13.26)$$

In addition to breaking  $\text{SO}(10)$  to  $\text{SO}(9)$ , this condensate  $\text{SU}(2)$  to a subgroup  $\text{U}(1) \subset \text{SU}(2)$ .

The  $2 \times 2 \rightarrow 3_s$  channel is actually a repulsive channel for the  $\text{SU}(2)$  interaction, with  $\Delta C_2 = -1/2$ . If the  $\text{SU}(2)$  gauge interaction is weak enough, this does not matter, but if it is moderately strong, although weaker than the  $\text{SO}(10)$  gauge interaction, it might prevent the condensate from forming. However, we assume that the  $\text{SO}(10)$  coupling is sufficiently strong at a given scale  $\mu$  so that this condensate does form.

Having analyzed the situation in which the  $\text{SO}(10)$  gauge coupling is strong and dominates over the  $\text{SU}(2)$  gauge coupling, we next analyze the opposite situation in which the  $\text{SU}(2)$  gauge coupling becomes sufficiently strong and dominates over the  $\text{SO}(10)$  coupling. The condensate then forms in the MAC for  $\text{SU}(2)$ , which is  $2 \times 2 \rightarrow 1_a$ , involving an antisymmetric contraction of  $\text{SU}(2)$  indices with the  $\epsilon_{\alpha\beta}$  tensor.

$$\langle \epsilon_{\alpha\beta} \psi_L^\alpha {}^T C \psi_L^\beta \rangle. \quad (13.27)$$

The general result (2.5) then implies that the relevant representation in the Clebsch-Gordan decomposition of the direct product  $\mathcal{S} \times \mathcal{S}$  is antisymmetric, and we therefore denote it as  $R_a$ . As discussed above, given that  $k$  is even here, the representation that would normally be favored as the MAC in the direct product of two

spinors, (13.12), namely the  $\square$  representation, is symmetric rather than antisymmetric, and hence  $R_a$  cannot be  $\square$ . Instead, the lowest-dimension representation in the expansion (13.12) that is odd under interchange of the spinors is  $A_3$  with dimension  $\binom{4k+2}{3}$ , so the condensation channel is

$$(\mathcal{S}, \square) \times (\mathcal{S}, \square) \rightarrow ((A_3)_a, 1_a). \quad (13.28)$$

The measure of attractiveness of this channel is given by the  $\Delta C_2$  in Eq. (13.19) and is always positive for  $k \geq 2$ . Explicitly, for our  $\text{SO}(10)$  Model B theory, the  $A_3$  representation has dimension 120. When expressed as a sum of product representations of various  $\text{SO}(10)$  subgroups, the 120-dimensional representation has no singlets under either of the maximal (i.e., rank-5) subgroups  $\text{SU}(5) \otimes \text{U}(1)$  and  $\text{SU}(4) \otimes \text{SU}(2) \otimes \text{SU}(2)$ , or the rank-4 subgroup  $\text{SO}(9)$ , but does contain a singlet under the rank-4 subgroup  $\text{SO}(7) \otimes \text{SU}(2)$  [23]. It therefore breaks  $\text{SO}(10)$  to  $\text{SO}(7) \otimes \text{SU}(2)$ .

### C. Model C

Here we analyze a model, denoted Model C, that has a fermion sector which is a combination of the fermion sectors of Model A in Section XIII A and Model B in Section XIII B, and thus is of the form  $\{f_{ns,s}, f_{ns,ns}\}$ . These fermions consist of  $n$  copies of the  $(\mathcal{S}, 1)$  fermion  $\omega_{i,L}$ ,  $i=1, \dots, n$ , as in Eq. (13.6) and a single copy of the  $(\mathcal{S}, \square)$  fermion,  $\psi_{1,L}^\alpha$ , as in Eq. (13.20).

The one-loop coefficient in the beta function of the  $\text{SU}(2)$  gauge interaction in this Model C is the same as (13.22) for Model B, and hence the requirement that the  $\text{SU}(2)$  gauge interaction must be asymptotically free restricts  $p \leq 1$ . The case  $p=0$  reduces to Model A, which we have already discussed. Therefore, as indicated, we take  $p=1$  here. This, in turn, restricts  $k$  to be equal to 2, i.e.,  $G_1 = \text{SO}(10)$ .

The one-loop coefficient in the  $\text{SO}(10)$  beta function for this Model C is

$$b_{1,\text{SO}(10)} = \frac{2}{3}(20-n), \quad (13.29)$$

so the asymptotic freedom of the  $\text{SO}(10)$  gauge interaction implies that  $n < 20$ .

If the  $\text{SO}(10)$  gauge interaction is sufficiently strong and dominates over the  $\text{SU}(2)$  interaction, then the resultant condensates include those analyzed for Models A and B above, together with a new type of condensate. This new condensate occurs in the channel

$$(\mathcal{S}, 1) \times (\mathcal{S}, \square) \rightarrow (\square, \square) \quad (13.30)$$

with corresponding condensate

$$\langle \omega_{i,L}^T C \psi_L^\alpha \rangle, \quad i \in \{1, \dots, n\}. \quad (13.31)$$

This condensate breaks  $\text{SO}(10)$  to  $\text{SO}(9)$ , which is vectorial and does not break further.

If, on the other hand, the  $SU(2)$  gauge interaction is sufficiently strong and dominates over the  $SO(10)$  interaction, then the condensate formation and symmetry-breaking is the same as for Model B, discussed in Section XIII B.

#### XIV. $SO(4k+2) \otimes SU(M)$ THEORY

Here we consider a chiral gauge theory with the gauge group

$$SO(N) \otimes SU(M), \text{ with } N = 4k + 2 = 2r \text{ and } M \geq 3. \quad (14.1)$$

We will show that the constraint of asymptotic freedom of both gauge interactions limits  $k$  to the single value  $k = 2$ , but in order to show this, we must first keep  $k \geq 2$  general. The fermion content is the sum over representations  $\mathcal{R}$  of

$$\begin{aligned} & \dim(\mathcal{R}_{SU(M)}) (\mathcal{R}_{SO(4k+2)}, 1) + (\bar{\mathcal{R}}_{SO(4k+2)}, \bar{\mathcal{R}}_{SU(M)}) \\ & + \dim(\mathcal{R}_{SO(4k+2)}) (1, \mathcal{R}_{SU(M)}) . \end{aligned} \quad (14.2)$$

In the classification of Section II), this theory is of the  $(cs, cav)$  type. We take  $M \geq 3$  since the theory with  $M = 2$  has a vectorlike subsector comprised of the  $(1, 2)$  fermions and is therefore not irreducibly chiral. Note that even if  $M = 2$ , this theory does not coincide with any of Models A, B, or C in Section XIII because those models also avoided  $(1, \square) = (1, 2)$  fermions that would have constituted a vectorlike subsector. However, if one were to take  $M = 2$ , then the  $SO(4k+2)$ -nonsinglet fermion sector would coincide with that of Model B in Section XIII. We will show below that  $M$  is limited to a finite set of values by the constraint of asymptotic freedom. For our present purposes, it will suffice to consider the simplest realization of this theory, with a single representation  $\mathcal{R}$  of  $SO(4k+2)$ , namely the smallest complex one, the spinor, and the smallest nonsinglet representation of  $SU(2)$ , namely the fundamental. The resultant fermion content is thus

$$p(\mathcal{S}, \square), \quad 2^{r-1}p(1, \bar{\square}). \quad (14.3)$$

The one-loop coefficient of the  $SO(4k+2)$  beta function (with  $4k+2 = 2r$ ) is

$$b_{SO(4k+2)} = \frac{2}{3} [11(r-1) - 2^{r-4}pM]. \quad (14.4)$$

The requirement that the  $SO(4k+2)$  gauge interaction must be asymptotically free then yields the upper bound

$$p < \frac{11(r-1)}{2^{r-4}M}. \quad (14.5)$$

Although we restrict  $M \geq 3$ , we note that if one were to take  $M = 2$ , then this would be the same as the upper bound (13.23) on  $p$  for Model B in Section XIII. The

fact that we take  $M \geq 3$  here makes this a more stringent upper bound than (13.23).

We denote the fermion fields for this theory as

$$\psi_{i,L}^\alpha, \quad i = 1, \dots, p: \quad p(\mathcal{S}, \square) \quad (14.6)$$

and

$$\eta_{\alpha,j,L}, \quad j = 1, \dots, 2^{r-1}p: \quad 2^{r-1}p(1, \bar{\square}), \quad (14.7)$$

where  $\alpha$  is an  $SU(M)$  gauge index and  $i, j$  are flavor indices.

The one-loop coefficient of the  $SU(M)$  beta function is

$$b_{SU(M)} = \frac{1}{3}(11M - 2^r p). \quad (14.8)$$

The requirement that the  $SU(M)$  gauge interaction must be asymptotically free then yields the upper bound

$$p < \frac{11M}{2^r}. \quad (14.9)$$

For the relevant range  $M \geq 3$ , these two asymptotic freedom constraints can only be satisfied for  $r$  equal to its minimal value,  $r = 5$ , i.e.,  $k = 2$  and  $G_1 = SO(10)$ ; furthermore, given that  $k = 2$ , there are only a finite set of pairs  $(M, p)$  that satisfy the constraints. For the two integer intervals  $3 \leq M \leq 5$  and  $11 \leq M \leq 21$ , only the value  $p = 1$  is allowed, while for  $6 \leq M \leq 10$ ,  $p$  may take on the values 1 or 2. If  $M \geq 22$ , there are no allowed solutions for  $p$ . Our general construction is thus reduced to the finite family of chiral gauge theories with the gauge groups  $SO(10) \otimes SU(M)$  with  $3 \leq M \leq 21$  and the aforementioned possible values of  $p$  as a function of  $M$ .

If the  $SO(10)$  gauge coupling becomes sufficiently large and dominates over the  $SU(M)$  gauge coupling, then the former can produce condensation in the  $SO(10)$  MAC, namely (13.14). Since the  $\square$  is symmetric under interchange of the spinors in (13.14) for even  $k$  and hence, in particular, for  $k = 2$ , i.e.,  $SO(10)$ , it follows from our general result (2.5) that the combination of the  $SU(M)$  and flavor product  $S_{ij}$  must be symmetric. For the values of  $M$ , namely  $3 \leq M \leq 5$  and  $11 \leq M \leq 21$  that allow only  $p = 1$ , it follows that the flavor product must be symmetric, as  $S_{ij} = S_{11}$  and hence that the channel is, in terms of the full representations,

$$(\mathcal{S}, \square) \times (\mathcal{S}, \square) \rightarrow (\square_s, \square) \quad (14.10)$$

with the condensate

$$\langle \psi_{1,L}^{\alpha T} C \psi_{1,L}^\beta \rangle \quad (14.11)$$

The  $SO(10)$   $\Delta C_2$  measure of attractiveness for this channel is given by the  $N = 10$  special case of Eq. (13.15), namely  $27/4$ . However, the  $SU(M)$   $\Delta C_2$  value is negative, as is evident from Eq. (5.20), setting  $M = N$ , so this is a repulsive channel as regards the  $SU(M)$  interaction. This breaks  $SO(10)$  to  $SO(9)$ , which is vectorial,

and does not break further. Using a vacuum alignment argument, one may infer that  $\alpha = \beta$  so that the condensate (14.11) breaks  $SU(M)$  to  $SU(M-1)$ .

For the interval  $6 \leq M \leq 10$  where the theory allows  $p = 2$ , the dynamics could instead produce a condensate in the channel

$$(\mathcal{S}, \square) \times (\mathcal{S}, \square) \rightarrow (\square_s, \square) \quad (14.12)$$

where the flavor product  $S_{ij}$  is antisymmetric, so that the condensate is

$$\langle \psi_{1,L}^\alpha C \psi_{2,L}^\beta - \psi_{2,L}^\alpha C \psi_{1,L}^\beta \rangle. \quad (14.13)$$

In addition to being attractive as regards the  $SO(10)$  interaction, the channel (14.12) is also attractive with respect to the  $SU(M)$  interaction, with  $\Delta C_2$  given by Eq. (5.19) with  $N = M$ . Hence, for  $M$  in the interval  $6 \leq M \leq 10$  where  $p = 2$  is allowed, we infer that the preferred condensation channel in the case where  $SO(10)$  is strong is (14.12). This breaks  $SO(10)$  to  $SO(9)$  and  $SU(M)$  to  $SU(M-2) \otimes SU(2)$ .

## XV. $SO(4k+2) \otimes SO(4k'+2)$ THEORY

Here we explore a chiral gauge group of the  $(cs, cs)$  type, in our classification from Section (II). For this purpose, we choose the gauge group

$$SO(4k+2) \otimes SO(4k'+2), \quad \text{where } k, k' \geq 2 \quad (15.1)$$

and fermion content consisting of  $p$  copies of the bi-spinor representation,  $(\mathcal{S}, \mathcal{S})$ . We set  $N = 4k+2 = 2r$  and  $N' = 4k'+2 = 2r'$ . Although this family of theories ostensibly depends on the three parameters  $k, k'$ , and  $p$ , we will show that there is only one allowed choice for these three parameters.

The one-loop coefficients in the  $SO(4k+2)$  and  $SO(4k'+2)$  beta functions are

$$b_{SO(4k+2),1} = \frac{2}{3} [11(r-1) - 2^{r+r'-5}p] \quad (15.2)$$

and

$$b_{SO(4k'+2),1} = \frac{2}{3} [11(r'-1) - 2^{r+r'-5}p]. \quad (15.3)$$

The requirements that the  $SO(4k+2)$  and  $SO(4k'+2)$  gauge interactions must be asymptotically free yield the upper bounds

$$p < \frac{11(r-1)}{2^{r+r'-5}} \quad (15.4)$$

and

$$p < \frac{11(r'-1)}{2^{r+r'-5}} \quad (15.5)$$

These can only be satisfied by the single set of values  $r = r' = 5$  and  $p = 1$ , i.e., for the group  $SO(10) \otimes SO(10)$

with  $p = 1$  copy of the  $(\mathcal{S}, \mathcal{S})$  fermion. The structure of this theory is thus symmetric under interchange of the two factor groups. If we break this symmetry by setting one  $\alpha_i$  to be large and the other small in Eq. (2.3), then we can obtain situations in which one  $SO(10)$  coupling dominates over the other. However, because of the structural symmetry, in contrast to the generic behavior that we have found for the other direct-product chiral gauge theories that we have investigated, here the pattern of symmetry breaking is the same regardless of which  $SO(10)$  gauge coupling is large and dominant.

If the first  $SO(10)$  gauge coupling gets large enough and dominates over the second  $SO(10)$  gauge coupling, or vice versa, this can produce fermion condensation in the channel

$$(\mathcal{S}, \mathcal{S}) \times (\mathcal{S}, \mathcal{S}) \rightarrow (\square_s, \square_s), \quad i.e.,$$

$$(16, 16) \times (16, 16) \rightarrow (10_s, 10_s) \quad (15.6)$$

where we have used the fact that  $k$  and  $k'$  are even to infer the symmetry properties of  $(\square, \square)$  in the Clebsch-Gordan decomposition of the direct product of the spinors. This condensation breaks the gauge symmetry  $SO(10) \otimes SO(10)$  to  $SO(9) \otimes SO(9)$ , which is vectorial and does not break further.

## XVI. $SU(N) \otimes SU(M)$ THEORY

### A. General Formulation

In this section we analyze a chiral gauge theory with a gauge group

$$G = SU(N) \otimes SU(M) \quad (16.1)$$

and fermion content consisting of a sum over  $\mathcal{R}_{SU(N)}$  and  $\mathcal{R}_{SU(M)}$  of

$$\begin{aligned} & \dim(\mathcal{R}_{SU(M)}) (\mathcal{R}_{SU(N)}, 1) + (\bar{\mathcal{R}}_{SU(N)}, \bar{\mathcal{R}}_{SU(M)}) \\ & + \dim(\mathcal{R}_{SU(N)}) (1, \mathcal{R}_{SU(M)}), \end{aligned} \quad (16.2)$$

where  $\mathcal{R}_{SU(N)}$  and  $\mathcal{R}_{SU(M)}$  denote representations of  $SU(N)$  and  $SU(M)$ , respectively. This theory is of type  $(cav, cav)$  in the classification of Section II. A special case of this theory with  $\mathcal{R}_{SU(N)}$  and  $\mathcal{R}_{SU(M)}$  both equal to the fundamental representation was studied before in [5, 6], but in both of these previous works, it was studied as an example of a preon theory that might confine without spontaneous symmetry breaking and hence produce massless composite fermions. Here we consider it in a different way, as a theory that can self-break with bilinear fermion condensate formation, and we study the generalized theory with fermion representations higher than the fundamental.

The numbers  $M \geq 2$  and  $N \geq 2$ , subject to the asymptotic freedom constraint (16.9) below. This is an irreducibly chiral gauge theory, so the chiral gauge invariance

precludes any mass terms in the fundamental Lagrangian of the theory. One easily checks that this theory is free of any anomalies in gauged currents. It is also free of any global anomalies in the case where  $N$  or  $M$  is equal to 2. To see this, consider, for example, the case where  $N = 2$  and the fermions that are nonsinglets under this group transform as doublets. From Eq. (16.2) one sees that the number of  $SU(2)$  doublets is  $2\dim(\mathcal{R}_{SU(M)})$  and hence is even.

We calculate the one-loop coefficients in the  $SU(N)$  and  $SU(M)$  beta functions to be

$$b_{1,SU(N)} = \frac{1}{3} \left[ 11N - 4 \dim(\mathcal{R}_{SU(M)}) T(\mathcal{R}_{SU(N)}) \right] \quad (16.3)$$

and

$$b_{1,SU(M)} = \frac{1}{3} \left[ 11M - 4 \dim(\mathcal{R}_{SU(N)}) T(\mathcal{R}_{SU(M)}) \right]. \quad (16.4)$$

Hence, the requirements that the  $SU(N)$  and  $SU(M)$  gauge interactions should be asymptotically free imply, respectively, that

$$\dim(\mathcal{R}_{SU(M)}) T(\mathcal{R}_{SU(N)}) < \frac{11N}{4} \quad (16.5)$$

and

$$\dim(\mathcal{R}_{SU(N)}) T(\mathcal{R}_{SU(M)}) < \frac{11M}{4}. \quad (16.6)$$

### B. Model with Fermions $(F, F)$

Here we consider the version of the general theory of type (16.1) containing fermions with  $\mathcal{R}_{SU(N)} = \square$  and  $\mathcal{R}_{SU(M)} = \square$  (an equivalent notation is  $F = \square$ ). Then

$$b_{1,SU(N)} = \frac{1}{3}(11N - 2M) \quad (16.7)$$

and

$$b_{1,SU(M)} = \frac{1}{3}(11M - 2N), \quad (16.8)$$

so the inequalities (16.5) and (16.6) read  $M < 11N/2$  and  $N < 11M/2$ , and the range of  $N$  and  $M$  allowed by these two constraints is given by

$$\frac{2}{11} < \frac{M}{N} < \frac{11}{2}. \quad (16.9)$$

We denote the fermion fields as

$$\omega_{i,L}^a, \quad i = 1, \dots, M : \quad M(N, 1), \quad (16.10)$$

$$\zeta_{a,\alpha,L} : \quad (\bar{N}, \bar{M}), \quad (16.11)$$

and

$$\eta_{j,L}^\alpha, \quad j = 1, \dots, N : \quad N(1, M), \quad (16.12)$$

where  $a$  and  $\alpha$  denote, respectively,  $SU(N)$  and  $SU(M)$  gauge indices and  $i \in \{1, \dots, M\}$  and  $j \in \{1, \dots, N\}$  are copy (flavor) indices.

As noted, one possibility is confinement without any spontaneous chiral symmetry breaking, leading to massless composite spin 1/2 fermions that are singlets under  $SU(N) \otimes SU(M)$ . We investigate here the alternative possibility of condensate formation and associated chiral symmetry breaking. If the  $SU(N)$  gauge interaction is sufficiently strong and dominates over the  $SU(M)$  interaction, then this  $SU(N)$  interaction can produce condensation in the most attractive channel  $N \times \bar{N} \rightarrow 1$ . For the full theory, this is the channel

$$(N, 1) \times (\bar{N}, \bar{M}) \rightarrow (1, \bar{M}), \quad (16.13)$$

with attractiveness measure given by  $\Delta C_2 = 2C_2(N) = (N^2 - 1)/N$ . The associated condensates are of the form

$$\langle \omega_{i,L}^{a,T} C \zeta_{a,\alpha,L} \rangle, \quad i = 1, \dots, M \quad (16.14)$$

(where the sum over  $a$  here and below is from  $a = 1$  to  $a = N$ ). Consider the condensate (16.14) with  $i = 1$ . Since this transforms as a  $\bar{M}$  representation of  $SU(M)$ , it breaks this symmetry to  $SU(M - 1)$ . By convention, we may use the initial  $SU(M)$  invariance to pick  $\alpha = M$  in this condensate, so that it is

$$\langle \omega_{1,L}^{a,T} C \zeta_{a,M,L} \rangle. \quad (16.15)$$

We denote the scale where this condensate forms as  $\Lambda$ . The fermions  $\omega_{1,L}^a$  and  $\zeta_{a,M,L}$  with  $1 \leq a \leq N$  involved in this condensate thus gain dynamical masses of order  $\Lambda$ , as do the  $2M - 1$  gauge bosons in the coset  $SU(M)/SU(M - 1)$ . In the resultant  $SU(N) \otimes SU(M - 1)$  chiral gauge theory, we consider the condensate (16.14) with  $i = 2$  and  $\alpha \in \{1, \dots, M - 1\}$ . Again, by convention, we may use the residual  $SU(M - 1)$  gauge invariance to pick  $\alpha = M - 1$  in this condensate, so that it is

$$\langle \omega_{2,L}^{a,T} C \zeta_{a,M-1,L} \rangle. \quad (16.16)$$

This preserves  $SU(N)$  and transforms like the conjugate fundamental representation of  $SU(M - 1)$ , thereby breaking  $SU(M - 1)$  to  $SU(M - 2)$ . This fermion condensation process continues with the formation of the condensates

$$\langle \omega_{i,L}^{a,T} C \zeta_{a,M-i+1,L} \rangle, \quad i \leq M, \quad (16.17)$$

breaking  $SU(M)$  completely. The last-enumerated condensate is  $\langle \omega_{M,L}^{a,T} C \zeta_{a,1,L} \rangle$ . Since all of the condensates of the form (16.14) have the same attractiveness measure,  $\Delta C_2$ , they are expected to form at approximately the same scale,  $\Lambda$ . All of the chiral fermions  $\omega_{i,L}^a$  and  $\zeta_{a,\alpha}$  with  $1 \leq i \leq M$ ,  $1 \leq a \leq N$ , and  $1 \leq \alpha \leq M$  are involved in these condensates and gain dynamical masses of order

$\Lambda$ , as do the full set of  $M^2 - 1$   $SU(M)$  gauge bosons. This leaves a theory with an  $SU(N)$  gauge invariance containing the  $N^2 - 1$   $SU(N)$  gauge bosons and a set of  $MN$  massless  $SU(N)$ -singlet fermions, namely the  $\eta_{j,L}^\alpha$  with  $1 \leq \alpha \leq M$  and  $1 \leq j \leq N$ . The  $SU(N)$  pure gluonic theory then forms a spectrum of  $SU(N)$ -singlet glueballs.

Clearly, if the  $SU(M)$  gauge interaction is sufficiently strong and dominates over the  $SU(N)$  gauge interaction, then the above discussion applies with the replacements  $M \leftrightarrow N$  and  $\omega_{i,L}^a \rightarrow \eta_{j,L}^\alpha$ . In this case, the  $SU(M)$  interaction breaks the  $SU(N)$  gauge symmetry completely, leaving the  $MN$  massless  $SU(M)$ -singlet fermions  $\omega_{i,L}^a$  with  $1 \leq a \leq N$  and  $1 \leq i \leq M$ . The  $SU(M)$  pure gluonic theory then forms a spectrum of  $SU(M)$ -singlet glueballs.

The version of the general theory with gauge group (16.1) and fermion representations  $\mathcal{R}_{SU(N)} = \square$  and  $\mathcal{R}_{SU(M)} = \bar{\square}$  exhibits the same properties as those that we have analyzed, with obvious changes, so we do not discuss it separately.

### C. Model with $(F, A_2)$

We next analyze a model with the gauge group (16.1) and fermion representations  $\mathcal{R}_{SU(N)} = \square$  and  $\mathcal{R}_{SU(M)} = \bar{\square}$ . Since  $\bar{\square} = \square$  for  $SU(M) = SU(3)$ , we restrict  $M \geq 4$ . For this model the general equations (16.3) and (16.4) read

$$b_{1,SU(N)} = \frac{1}{3}[11N - M(M-1)] \quad (16.18)$$

and

$$b_{1,SU(M)} = \frac{1}{3}[11M - 2N(M-2)] . \quad (16.19)$$

The general inequalities (16.5) and (16.6) guaranteeing the asymptotic freedom of the  $SU(N)$  and  $SU(M)$  gauge interactions read, respectively,

$$N > \frac{M(M-1)}{11} \quad (16.20)$$

and

$$N < \frac{11M}{2(M-2)} . \quad (16.21)$$

In Fig. 2 we show a plot of the corresponding curves, as a function of  $M$ . The lower bound on  $N$  from (16.20) is  $N > 2$  for  $M = 4$  and increases as  $M$  increases. The upper bound on  $N$  from (16.21) is  $N < 11$  for  $M = 4$  and decreases as  $M$  increases. The curves for the upper and lower bounds on  $N$  as a function of  $M$  cross each other at

$$M = \frac{3 + 9\sqrt{3}}{2} = 9.294 \quad (16.22)$$

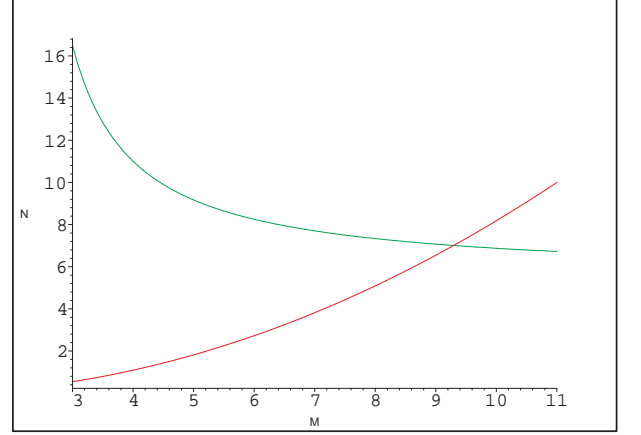


FIG. 2: Plot of the region in  $M$  and  $N$  allowed by the requirement of asymptotic freedom for the  $SU(N)$  and  $SU(M)$  gauge interactions in the  $SU(N) \otimes SU(M)$  Model with the  $(F, A_2)$  fermion content. The allowed values of  $M$  and  $N$  lie between the two curves. See text for further discussion.

where

$$N = \frac{123 + 18\sqrt{3}}{22} = 7.008 , \quad (16.23)$$

where the floating point values are given to the indicated accuracy. The allowed values of  $M$  and  $N$  thus lie within the enclosed region between the upper and lower curves in Fig. 2. This region has finite area and hence there are only finitely many allowed values of  $M$  and  $N$ . This is in contrast to the joint asymptotic freedom constraint for the model with  $(F, F)$  fermions, (16.9), which is an infinite wedge-shaped region in the  $M, N$  plane. As is evident, for a given  $M \geq 4$ , the range of allowed values of  $N$  decreases with increasing  $M$ . For  $M = 4$ ,  $N$  may take on values in the range  $2 \leq N \leq 10$ , while for  $M = 8$ , the allowed values of  $N$  are  $N = 6, 7$ , and for  $M = 9$ , there is only one allowed value of  $N$ , namely  $N = 7$ . If  $M \geq 10$ , there are no values of  $N$  that satisfy the inequalities (16.20) and (16.21).

## XVII. CONCLUSIONS

In summary, in this paper we have analyzed patterns of dynamical gauge symmetry breaking using a variety of chiral gauge theories with direct-product gauge groups containing asymptotically free non-Abelian gauge interactions of both unitary and orthogonal types. Our results on the strong-coupling behavior of these theories show that these patterns of symmetry breaking are typ-

ically quite different depending on the structure of the factor groups in the direct product and on which gauge interaction dominates in the formation of fermion condensates. These theories provide useful theoretical laboratories demonstrating explicitly the generic behavior that if the gauge coupling for one of the factor groups  $G_i \subset G$  gets sufficiently strong and dominates over the other(s), then it can produce bilinear fermion condensates that can self-break the  $G_i$  symmetry itself and/or break other gauge symmetries  $G_j \subset G$ . If the  $G_i$  gauge interaction that is dominant is vectorial, then it does not self-break, although it typically still breaks other gauge symmetries in the direct-product group. The theories that we have studied also yield useful examples of sequential gauge symmetry breaking. These results further elucidate the behavior of strongly coupled chiral gauge theories and are of value in extending the understanding of nonperturbative behavior of quantum field theories.

### Acknowledgments

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### Appendix A: Some Relevant Group Invariants

For reference, we list some group invariants here. We first define some notation. Let us denote the generators of the associated Lie algebra  $\mathfrak{g}$  as  $T_a$ , where  $a = 1, \dots, o(G)$ , where  $o(G)$  is the order of the group. These generators satisfy the commutation relation  $[T_a, T_b] = i c_{abd} T_d$ , where  $c_{abc}$  are the structure constants. For a representation  $\mathcal{R}$ , the Casimir invariants  $C_2(\mathcal{R})$  and  $T(\mathcal{R})$  are defined as

$$\sum_{i,j=1}^{\dim(\mathcal{R})} \mathcal{D}_{\mathcal{R}}(T_a)_{ij} \mathcal{D}_{\mathcal{R}}(T_b)_{ji} = T(\mathcal{R}) \delta_{ab} \quad (\text{A1})$$

and

$$\sum_{a=1}^{o(G)} \sum_{j=1}^{\dim(\mathcal{R})} \mathcal{D}_{\mathcal{R}}(T_a)_{ij} \mathcal{D}_{\mathcal{R}}(T_a)_{jk} = C_2(\mathcal{R}) \delta_{ik}, \quad (\text{A2})$$

where  $T_a$  are the generators of  $G$ , and  $\mathcal{D}_{\mathcal{R}}$  is the matrix representation (*Darstellung*) of  $\mathcal{R}$ . These satisfy

$$T(\mathcal{R}) o(G) = C_2(\mathcal{R}) \dim(\mathcal{R}), \quad (\text{A3})$$

where  $\dim(\mathcal{R})$  is the dimension of the representation  $\mathcal{R}$ .

For an  $SU(N)$  group, the rank is  $N - 1$  and group invariants (with the normalization convention  $\text{Tr}(T_a T_b) = (1/2) \delta_{ab}$ ) include the following (e.g., [23, 24])

$$C_2(\square) = \frac{N^2 - 1}{2N}, \quad (\text{A4})$$

$$C_2(\square\square) = \frac{(N+2)(N-1)}{N}, \quad (\text{A5})$$

and

$$C_2(\boxplus) = \frac{(N-2)(N+1)}{N}. \quad (\text{A6})$$

The rank of  $SO(N)$  is the integral part of  $N/2$ . We denote  $A_t$  the rank- $t$  antisymmetric tensor representation, with dimension  $\binom{N}{t}$ , where  $\binom{a}{b} = a!/[b!(a-b)!]$ . Note that for  $SO(N)$ , the adjoint representation is the same as  $A_2$  and the vector, fundamental, and  $A_1$  representations are the same. With an appropriate normalization convention for the generators of  $SO(N)$  (which does not affect the physics), one has [23, 24]

$$T(\text{adj}) = C_2(\text{adj}) = N - 2, \quad (\text{A7})$$

$$T(\square) = 1, \quad (\text{A8})$$

and

$$C_2(\square) = \frac{N-1}{2}. \quad (\text{A9})$$

For  $SO(N)$  with  $N = 2r$  and  $\mathcal{S}$  the spinor representation,

$$\dim(\mathcal{S}) = 2^{r-1} \quad (\text{A10})$$

$$T(\mathcal{S}) = 2^{r-4} \quad (\text{A11})$$

$$C_2(\mathcal{S}) = \frac{r(2r-1)}{8}. \quad (\text{A12})$$

Denoting the antisymmetric rank- $t$  tensor representation of  $SO(2r)$  as  $A_t$ , one has

$$C_2(A_t) = \frac{t(2r-t)}{2}. \quad (\text{A13})$$

From the structure of the triangle diagram, it follows that triangle anomaly in gauged currents is proportional to

$$\text{Tr}(\mathcal{D}_{\mathcal{R}}(T_a), \{\mathcal{D}_{\mathcal{R}}(T_b), \mathcal{D}_{\mathcal{R}}(T_c)\}) = d_{abc} A_{\mathcal{R}} \quad (\text{A14})$$

Groups for which  $A_{\mathcal{R}} = 0$  include those with real or pseudoreal representations,  $SO(4k+2)$  for  $k \geq 2$ , and  $E_6$  [22, 23]. For the symmetric and antisymmetric rank- $t$  tensor representations of  $SU(N)$ , the anomaly is, respectively [22]

$$\mathcal{A}(S_t) = \frac{(N+t)!(N+2t)}{(N+2)!(t-1)!}. \quad (\text{A15})$$

and, for  $1 \leq t \leq N-1$ ,

$$\mathcal{A}(A_t) = \frac{(N-3)!(N-2t)}{(N-t-1)!(t-1)!}. \quad (\text{A16})$$

In particular,  $\mathcal{A}(S_2) = N + 4$  and  $\mathcal{A}(A_2) = N - 4$ .

A gauge theory in  $d = 4$  dimensions with gauge group  $G$  contains instantons if  $\pi_{d-1}(G) = \pi_3(G)$  is nontrivial. One has [27]

$$\pi_3(\mathrm{SU}(N)) = \mathbb{Z} \quad (\mathrm{A17})$$

and

$$\pi_3(\mathrm{SO}(N)) = \mathbb{Z} \quad \text{if } N \geq 5. \quad (\mathrm{A18})$$

The global anomaly in an  $\mathrm{SU}(2)_L$  gauge theory is due to

$$\pi_4(\mathrm{SU}(2)) = \mathbb{Z}_2 \quad (\mathrm{A19})$$

Further,

$$\pi_4(\mathrm{SU}(N)) = \emptyset \quad \text{if } N \geq 3 \quad (\mathrm{A20})$$

and

$$\pi_4(\mathrm{SO}(N)) = \emptyset \quad \text{if } N \geq 6. \quad (\mathrm{A21})$$

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